Pseudocharacters of Classical Groups

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Abstract

A pseudocharacter is a function from a group to a ring satisfying polynomial relations which make it "look like" the character of a representation. The key feature of pseudocharacters is that when the ring is an algebraically closed field, pseudocharacters are identical to true characters. Recent work by Vincent Lafforgue suggests that pseudocharacters can be defined for arbitrary connected reductive groups, such that pseudocharacters for a specific group are in bijection with semisimple equivalence classes of representations having image in that group. In this project, we refine Lafforgue's result by showing that pseudocharacters consisting of finitely many functions exist for any connected reductive group, although the proof is non-constructive. We then use classical invariant theory to explicitly define pseudocharacters for the orthogonal, symplectic, general orthogonal, and general symplectic groups. We apply these results to make partial progress towards constructing universal deformation rings for orthogonal and symplectic representations. We also use pseudocharacters to investigate the question of local vs. global conjugacy on the Galois side of the Langlands correspondence.

1 Introduction

The study of Galois representations – functions which describe the structure of Galois groups, which in turn describe the structure of systems of number-theoretic equations – forms a large and active part of modern number theory. In particular, much attention has been paid to the Langlands program over the past 50 years. The Langlands program is a broad set of conjectures that relate Galois representations to automorphic representations, which describe concrete, smooth functions relating to matrix groups H. The idea is that by relating these concepts, we can deduce concrete properties of number-theoretic equations (which are quite challenging to study) from concrete properties of special smooth functions. For an introduction and statements of the Langlands conjectures, see, e.g., Gelbart[1] or Kudla et. al.[2] When $H = GL_1(\mathbb{C}) = \mathbb{C}^\times$ consists of all invertible 1×1 matrices over the complex numbers, i.e., all non-zero complex numbers, the Langlands conjectures reduce to class field theory, an important part of early 20th century number theory. When $H = GL_2(\mathbb{C})$, the Langlands conjectures tie in to the proof of Fermat's Last Theorem.

The idea of a *pseudocharacter* appears in several places in existing work on Galois representations. A character is a complex-valued function defined in terms of a Galois representation which is simpler than the original representation but still describes most of its properties. Intuitively, a pseudocharacter is a function which "looks like" a character of a Galois representation, in the sense that it satisfies some equations which characters also satisfy. Pseudocharacters were introduced for GL_2 (2×2 invertible matrices) by Wiles[3] and for GL_d ($d\times d$ invertible matrices) by Taylor[4] (they use the name "pseudo-representations"). Using classical invariant theory,[5] which studies functions on matrices and other vector spaces that do not change under specified changes of coordinates, Taylor proved that every complex-valued pseudocharacter is an actual character. He used this result to construct specific Galois representations taking values in GL_d, a problem which is in general very difficult.

Pseudocharacters appear in a slightly different form in recent work by Vincent Lafforgue,[6] in which he establishes one part of the Langlands program for many matrix groups H over global function fields (analogs of the rational numbers that are often easier to study). Specifically, he shows how to construct Galois representations from automorphic representations. One ingredient (among many) in his proof is Proposition 11.7,[6] which embodies a general method for defining pseudocharacters of some matrix groups based on their invariant theory. (By a pseudocharacter "of" a matrix group, I mean a set of functions satisfying certain simple equations, such that those functions correspond in a natural way to Galois representations for H.) Indeed, when $H = GL_d$, Lafforgue's result, coupled with Procesi's invariant theory, easily implies Taylor's work on pseudocharacters (see Remark 11.8[6]).

Lafforgue's result strongly suggests that one should be able to define pseudocharacters of matrix groups besides GL_d, especially the classical groups O_d (orthogonal group), SO_d (special orthogonal group), and Sp_{2d} (symplectic group), since their invariant theory has been studied before.

In this project, we succeed in defining pseudocharacters for these groups and the related groups GO_d (general orthogonal group) and GSp_{2d} (general symplectic group). We also show that in principle, using Lafforgue's result, "nice" pseudocharacters consisting of finite amounts of data can be defined for a large class of matrix groups, the connected reductive groups.

Then, we use our pseudocharacters to investigate the problem of element-conjugacy vs. conjugacy for Galois representations. This problem, which has been investigated by previous authors[7][8],[9] essentially asks whether two Galois representations which are equivalent at each element are also equivalent as a whole. Our pseudocharacters allow us to give simpler and more general answers than those already known, at least for the classical groups mentioned above. Doing so sheds light on the precise form that the Langlands conjectures for those groups should take.

2 Results

2.1 Pseudocharacters of Specific Groups

Taylor's result on GL_d-pseudocharacters is as follows. Let G be a group and A be a commutative ring with identity. Define a GL_d-pseudocharacter of G over A to be a set map $T : G \to A$ such that

- $T(1) = n$

- For all $g_1, g_2 \in G$, $T(g_1 g_2) = T(g_2 g_1)$
- For all $g_1, \ldots, g_{n+1} \in G$,
$$\sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) T_\sigma(g_1, \ldots, g_{n+1}) = 0,$$
where S_{n+1} is the symmetric group on $n+1$ letters, $\mathrm{sgn}(\sigma)$ is the permutation sign of σ, and T_σ is defined by
$$T_\sigma(g_1, \ldots, g_{n+1}) = T(g_{i_1^{(1)}} \cdots g_{i_{r_1}^{(1)}}) \cdots T(g_{i_1^{(s)}} \cdots g_{i_{r_s}^{(s)}})$$
when σ has cycle decomposition $(i_1^{(1)} \ldots i_{r_1}^{(1)}) \ldots (i_1^{(s)} \ldots i_{r_s}^{(s)})$.

If T is a pseudocharacter, then define the kernel of T by
$$\ker(T) = \{h \in G | T(gh) = T(g) \text{ for all } g \in G\}$$

Then:

Theorem 2.1.1 (Theorem 1[4]). 1. *Let $\rho : G \to GL_d(A)$ be a representation. Then $\mathrm{Tr}(\rho)$ is a GL_d-pseudocharacter.*

2. *Suppose A is a field of characteristic 0, and let $\rho : G \to GL_d(A)$ be a representation. Then $\ker(\mathrm{Tr}(\rho)) = \ker(\rho^{ss})$, where ρ^{ss} denotes the semisimplification of ρ.*

3. *Suppose A is an algebraically closed field of characteristic 0. Let $T : G \to A$ be a GL_d-pseudocharacter. Then there is a semisimple representation $\rho : G \to GL_d(A)$ such that $\mathrm{Tr}(\rho) = T$, unique up to conjugation.*

4. *Suppose G is finitely generated. Then there is a finite subset $S \subset G$ such that for any $\mathbb{Z}[1/d!\,]$-algebra A, if $T : G \to A$ is a GL_d-pseudocharacter, then T is determined by its values on S.*

5. *If G and A are taken to be topological, then the above statements hold in topological/continuous form.*

We have derived results very similar to these in the case that the target group GL_d is replaced by an orthogonal group O_d, general orthogonal group GO_d, symplectic group Sp_{2d}, general symplectic group GSp_{2d}, or special orthogonal group SO_d. The main difference between the various groups is in the definition of their pseudocharacters.

First, we state the result for the general orthogonal group. Again let G be a group and let A be a commutative ring with identity. Recall that the general orthogonal group over A of dimension d is defined by
$$GO_d(A) := \{A \in M_d(A) | AA^t = \lambda I \text{ for some } \lambda \in A^*\}$$
Here A^* denotes the unit group of A.

A *GO_d-pseudocharacter of G over A* is a pair (T, λ), consisting of a set map $T : G \to A$ and a group homomorphism $\lambda : G \to A^*$, such that:

- $T(1) = d$
- For all $g_1, g_2 \in G$, $T(g_1 g_2) = T(g_2 g_1)$

- For all $g \in G$, $T(g) = \lambda(g)T(g^{-1})$
- For all integers $0 \leq k \leq (d+1)/2$ and for all $g_1, \ldots, g_{d+1} \in G$, T satisfies the relation $F^{\lambda}_{k,d+1}(T, g_1, \ldots, g_{d+1})$, where $F^{\lambda}_{k,d+1}$ is the function $F_{k,d+1}$ used in Theorem 8.4(a)[5] except that we replace Tr with T, and we replace any inverted element h^{-1} with $\lambda(h)h^{-1}$ and then distribute the scalars $\lambda(h)$ outside of any application of T.

Note that T is always a GL_d-pseudocharacter, since $F^{\lambda}_{0,d+1}$ is the nontrivial GL_d-pseudocharacter relation.

The kernel of a GO_d-pseudocharacter (T, λ) is

$$\ker(T, \lambda) := \ker(T) \cap \ker(\lambda).$$

Given a representation $\rho : G \to GO_d(A)$, define a set map $\lambda(\rho) : G \to A^*$ by letting $\lambda(g)$ be the scalar such that $\rho(g)\rho(g)^t = \lambda(g)I$. Easily $\lambda(\rho)$ is a group homomorphism.

Then we have the following result:

Theorem 2.1.2. *1. Let $\rho : G \to GO_d(A)$ be a general orthogonal representation. Then $(\mathrm{Tr}(\rho), \lambda(\rho))$ is a GO_d-pseudocharacter.*

2. *Suppose A is a field of characteristic 0, and let $\rho : G \to GO_d(A)$ be a general orthogonal representation. Then $\ker(\mathrm{Tr}(\rho), \lambda(\rho)) = \ker(\mathrm{Tr}(\rho)) = \ker(\rho^{ss})$.*
3. *Suppose A is an algebraically closed field of characteristic 0. Let $T : G \to A$ be a GO_d-pseudocharacter. Then there is a semisimple general orthogonal representation $\rho : G \to GO_d(A)$ such that $\mathrm{Tr}(\rho) = T$ and $\lambda(\rho) = \lambda$, unique up to conjugation by $GO_d(A)$.*
4. *Suppose G is finitely generated. Then there is a finite subset $S \subset G$ such that for any $\mathbb{Z}[1/d!\,]$-algebra A, if (T, λ) is a GO_d-pseudocharacter of G over A, then T is determined by its values on A.*
5. *If G and A are taken to be topological, then the above statements hold in topological/continuous form.*

Next, we define an O_d-pseudocharacter to be a set map $T : G \to A$ such that $(T, 1)$ is a GO_d-pseudocharacter. Specifying $\lambda = 1$ in the above theorem, we easily derive an analogous theorem with GO_d replaced by O_d.

The general symplectic and symplectic cases are identical, except with a different collection of relations to replace $F_{k,d+1}$, namely, the relations $F^i_{h,d}$ from Theorem 10.2(a).[5]

Finally, we can define SO_d-pseudocharacters as well, at least when A is a field of characteristic 0. For fixed d, it is well-known how to write $\det(M)$ as a $\mathbb{Z}[1/d!\,]$-polynomial in terms of $\mathrm{Tr}(M), \mathrm{Tr}(M^2), \ldots, \mathrm{Tr}(M^d)$ for $M \in M_d(A)$. Given an O_d-pseudocharacter $T : G \to A$, let $\det(T)(g)$ denote the natural application of this polynomial to $T(g), \ldots, T(g^d)$ for $g \in G$. When d is odd, we define a SO_d-pseudocharacter to be an O_d-pseudocharacter which additionally satisfies the relation $\det(T)(g) = 1$ for all $g \in G$. Then the same result holds as in the orthogonal case, with O_d replaced by SO_d.

When d is even, we define a SO_d-pseudocharacter of G over A to be a pair of functions $T : G \to A$, $P : G^{d/2} \to A$, such that:

- T is an O_d-pseudocharacter of G over A
- For all $g \in G$, $\det(T)(g) = 1$
- For all $g_1, \ldots, g_{d/2}, h_1, \ldots, h_{d/2} \in G$, $P(g_1, \ldots, g_{d/2})P(h_1, \ldots, h_{d/2})$ satisfies the relation in Theorem 3.2[10] with P in place of Q and T in place of Tr.

The kernel of a pseudocharacter (T, P) is defined to be the usual kernel of T.

Given a representation $\rho : G \to SO_d(A)$, we define a function $\mathrm{pl}(\rho) : G^{d/2} \to A$ by setting $\mathrm{pl}(\rho)(g_1, \ldots, g_{d/2}) = \mathrm{pl}(\rho(g_1), \ldots, \rho(g_{d/2}))$. Here $\mathrm{pl} : M_d^{d/2}(A) \to A$ is as defined by Aslaksen et. al.;[11] it is the full polarization of the antisymmetrized Pfaffian $\mathrm{pf}(M - M^t)$. Then a result completely analogous to the previous ones holds, where the pseudocharacter associated to a representation $\rho : G \to SO_d(A)$ is given by $(\mathrm{Tr}(\rho), \mathrm{pl}(\rho))$.

2.2 General Existence of Pseudocharacters

Let H be a linear algebraic group. Let G be any group, and let A be an algebraically closed field of characteristic 0. Modeling after the pseudocharacters above, we would like to define an H-pseudocharacter of a group G over a ring A to be a collection of functions $\{T^1_\alpha : G \to A\}_{\alpha \in I_1} \cup \{T^2_\alpha : G^2 \to A\}_{\alpha \in I_2} \cup \cdots$, satisfying some collection of polynomial relations whose inputs (which are plugged into the T^i_α) vary over all tuples of elements of G. These functions and relations should be chosen so that there is a natural bijection between H-pseudocharacters and between H-conjugacy classes of semisimple relations $G \to H$.

When H is a connected reductive algebraic group, Proposition 11.7[6] shows that H-pseudocharacters of this form do exist. (Note that Lafforgue's proof works equally well when Γ is an arbitrary group, and when E is replaced by any field of characteristic 0.) However, these pseudocharacters consist of infinitely many functions and hence cannot be made explicit. Using finiteness theorems of classical invariant theory, though, we prove that finitely many functions suffice, since their values determine the values of the rest.

2.3 Element-conjugacy and Conjugacy of Representations

Let H be a linear algebraic group, let G be an arbitrary group, and let $\rho, \rho' : G \to H$ be semisimple representations. Suppose that for each $g \in G$, $\rho(g)$ is conjugate to $\rho'(g)$ by an element of H. An interesting question to ask is: for what groups H do we then know that ρ is globally conjugate to ρ' by an element of H? For instance, when $H = GL_d$ is a general linear group, this question has an affirmative answer, by the classical fact that a representation is determined up to GL_d-conjugacy by its trace.

This question was investigated for finite G by Larsen[7],[8] where he gives an answer for many reductive groups H. Negative answers carry over immediately to arbitrary groups, but affirmative answers do not necessarily carry over. However, Larsen's classification has been mostly extended to the case of continuous representations of compact G by Fang et. al.[9]

Using our pseudocharacters, it is easy to extend these affirmative answers to the case of semisimple representations of arbitrary G for several reductive groups. This is because whenever H-

pseudocharacters can be defined to consist entirely of one-argument functions $G \to A$, semisimple element-conjugate representations $\rho, \rho' : G \to H$ have the same H-pseudocharacters, hence are globally conjugate by H. In particular, the orthogonal group, general orthogonal group, symplectic group, and general symplectic group have affirmative answers.

We also use pseudocharacter-like ideas and Larsen's original argument for finite G (Lemma 2.7[7]) to give an affirmative answer for the semidirect product of $SU(3)$ with its complex conjugation automorphism, which allows Proposition 2.8[7] to extend to continuous representations of compact G.

Finally, we use our pseudocharacters of the special orthogonal group to describe those special orthogonal representations having companions that are element-conjugate but not globally conjugate. Using this description, we construct a simpler example than the one used in Proposition 3.8,[7] in fact one with $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. The same example lets us extend Larsen's negative answer for $SO_{2d}(\mathbb{C})$ ($d \geq 4$) down to $SO_6(\mathbb{C}) \cong SL_4(\mathbb{C})/\{\pm 1\}$.

2.4 Future Work

It appears that our methods should allow us to give affirmative answers to the element-conjugacy vs. global conjugacy problem in two more cases: $G_2(\mathbb{C})$ for arbitrary domain groups, and $GL_d(\mathbb{C}) \rtimes \langle \tau \rangle$, where τ is the inverse-transpose outer automorphism. In principle, affirmative answers could be proven in both cases by showing that the invariants of these groups acting on themselves are generated by one-input functions. However, it is not yet clear how to do this, as the invariant theory of $GL_d(\mathbb{C}) \rtimes \langle \tau \rangle$ is not known, and the invariant theory of G_2[12] is rather complicated.

3 Methods

3.1 Pseudocharacters of Specific Groups

Let A be a commutative ring. Let R be a *-algebra over A, i.e., an A-algebra equipped with an A-linear map * satisfying $(r_1 r_2)^* = {r_2}^* {r_1}^*$. Let d be a positive integer.

An *orthogonal pseudocharacter of R over A of dimension* d is a set map $T : R \to A$ such that:

- $T(1) = d$
- For all $r_1, r_2 \in R$, $T(r_1 r_2) = T(r_2 r_1)$
- For all $r \in R$, $T(r) = T(r^*)$
- For all integers $0 \leq k \leq (d+1)/2$ and for all $r_1, \ldots, r_{d+1} \in R$, T satisfies the relation $F_{k,d+1}(T, r_1, \ldots, r_{d+1})$, where $F_{k,d+1}$ is the relation in Theorem 8.4(a)[5] with Tr replaced by T.

In turns out that $F_{0,d}$ is the nontrivial GL_d-pseudocharacter relation, so every orthogonal pseudocharacter is also a GL_d-pseudocharacter.

As in the GL_d case (but for algebras instead of groups), the kernel of T is

$$\ker(T) := \{r \in R| \text{ for all } s \in R,\, T(rs) = 0\}$$

Easily $\ker(T)$ is a 2-sided *-ideal of R, and we get an induced orthogonal pseudocharacter $T : R/\ker(T) \to A$.

An *orthogonal representation* of R of dimension d is an A-algebra morphism $\rho : R \to M_d(A)$ which takes * to the transpose.

Our main result about orthogonal pseudocharacters is essentially identical to the A-algebra version of Taylor's result on GL_d-pseudocharacters (Theorem 1[4]):

Theorem 3.1.1. *1. Let $\rho : R \to M_d(A)$ be an orthogonal representation. Then $\mathrm{Tr}(\rho)$ is an orthogonal pseudocharacter of dimension d.*

2. *Suppose A is a field of characteristic 0, and let $\rho : R \to M_d(A)$ be an orthogonal representation. Then $\ker(\mathrm{Tr}(\rho)) = \ker(\rho^{ss})$.*

3. *Suppose A is an algebraically closed field of characteristic 0. Let $T : G \to A$ be an orthogonal pseudocharacter of dimension d. Then there is a semisimple orthogonal representation $\rho : R \to M_d(A)$ such that $\mathrm{Tr}(\rho) = T$, unique up to conjugation by $O(d, A)$.*

4. *Suppose R is finitely generated as an A-algebra. Then there is a finite subset $S \subset R$ such that for any $\mathbb{Z}[1/d!\,]$-algebra R, if $T : R \to A$ is an orthogonal pseudocharacter of dimension d, then T is determined by its values on S.*

5. *If R and A are taken to be topological, then the above statements hold in topological/continuous form.*

Proof For consistency with the GL_d case, we will use O_d to denote the "abstract" orthogonal group, and $O_d(A)$ to denote the orthogonal group over A.

Parts (2) and (4) follow immediately from the corresponding results in the GL_d case, noting that the semisimplification of an orthogonal representation is again orthogonal by Theorem 15.2(b).[5]

By the same argument as in the GL_d case (see pp. 286-7[4]), to prove parts (1), (3), and (5), it suffices to prove parts (1) and (3) when R is a finitely generated free *-algebra and A is an algebraically closed field of characteristic 0. To prove parts (1) and (3) in this case, let $R = A[x_1, \ldots, x_r, y_1, \ldots, y_r]$, with * defined by $x_i* = y_i$. Let M_d^r be the affine variety consisting of r copies of M_d, and let the group variety O_d act on M_d^r by simultaneous conjugation. Then the orthogonal representations of R over A are in bijection with the points of $M_d^r(A)$ via the map $\rho \mapsto (\rho(x_1), \ldots, \rho(x_r))$.

Let $M_d^r//O_d := \mathrm{Spec}(A[M_d^r]^{O_d(A)})$ be the affine geometric quotient of M_d^r by O_d, which is an affine variety defined over A. By the theory of reductive groups and the fact that A is algebraically closed, we get a natural surjection

$$\phi : M_d^r(A) \twoheadrightarrow (M_d^r//O_d)(A)$$

By Theorem 15.3,[5] when ϕ is restricted to semisimple orthogonal representations, we get a bijection $\phi|_{ss}$ between semisimple orthogonal representations and $(M_d^r//O_d)(A)$.

By another result of Procesi, we can describe $A[M_d^r]^{O_d(A)}$ as follows.

Proposition 3.1.2 (Theorem 7.3(a)[5]). *Let $\{X_i, X_i^t\}^+$ denote the free semigroup on the $2r$ letters $X_1, \ldots, X_r, X_1^t, \ldots, X_r^t$. For any $w \in \{X_i, X_i^t\}^+$, let $T_w \in A[M_d^r]$ be the function given by $T_w(X_1, \ldots, X_r) = \mathrm{Tr}(w)$, where in $\mathrm{Tr}(w)$ we replace each letter with the matrix or transposed matrix of the same name. Then:*

(a) $A[M_d^r]^{O_d(A)}$ is generated by the functions T_w (in fact, by the functions U_w with w a word of length $\leq 2^d - 1$).

(b) The kernel of the natural map $\psi : A[\{Y_w | w \in \{X_i, X_i^t\}^+\}] \to A[M_d^r]^{O_d(A)}$ which sends Y_w to T_w is the radical of the ideal generated by the relations:

(R1) $Y_{w_1 w_2} = Y_{w_2 w_1}$, for all $w_1, w_2 \in \{A_i, A_i^t\}^+$.

(R2) $Y_w = Y_{w^t}$, for all $w \in \{X_i, X_i^t\}$, where w^t is defined as the transpose of w in the obvious way.

(R3) $F_{k,d+1}(Y, w_1, \ldots, w_{d+1}) = 0$, for all $0 \leq k \leq (d+1)/2$ and all $w_1, \ldots, w_{d+1} \in \{X_i, X_i^t\}^+$, where in the definition of $F_{k,d+1}^t$ we take $Y(w)$ to mean Y_w.

Thus taking the T_w as coordinate functions for $M_d^r//O_d$, we see that orthogonal pseudocharacters of R over A are in bijection with $(M_d^r//O_d)(A)$, via the map which takes T to the point $x \in (M_d^r//O_d)(A)$ with coordinates $T_w(x) = T(w)$. Then ϕ gives a surjection from orthogonal representations of R over A to orthogonal pseudocharacters of R over A, defined concretely by taking the trace. This proves part (1). Likewise, $\phi|_{ss}$ gives a bijection between semisimple orthogonal representations and orthogonal pseudocharacters. This proves part (3).

3.1.1 Symplectic Representations of *-algebras

The symplectic case is almost entirely identical to the orthogonal case, except that the transpose involution on $M_d(A)$ is replaced with symplectic involution:

$$X^* := \Omega^{-1} A^T \Omega,$$

where

$$\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

is the matrix of the standard symplectic form. The generators for the invariants are the same, so our pseudocharacter data is still just a function $T : R \to A$ representing the trace; however, the relations between the invariants are different.

Specifically, a *symplectic pseudocharacter of R over A of dimension $2d$* is a set map $T : R \to A$ such that:

- $T(1) = 2d$
- For all $r_1, r_2 \in R$, $T(r_1 r_2) = T(r_2 r_1)$
- For all $g \in R$, $T(r) = T(r^*)$
- For all integers $1 \leq i \leq d+1$ and $0 \leq h < i$, and for all $r_1, \ldots, r_{d+i} \in G$, T satisfies the relation $F_{h,d}^i(T, r_1, \ldots, r_{d+i}) = 0$, where $F_{h,d}^i$ is the relation in Theorem 10.2(a)[5] with Tr replaced by T.

It turns out that $F_{0,d}^{d+1}$ is the nontrivial GL_{2d}-pseudocharacter relation, so every symplectic pseudocharacter is also a GL_{2d}-pseudocharacter.

The kernel of T is again

$$\ker(T) := \{r \in R | \text{ for all } s \in R, T(rs) = 0\}$$

A *symplectic representation* of R of dimension $2d$ is an A-algebra morphism $\rho : R \to M_{2d}(A)$ which takes * to symplectic involution.

The main result is completely analogous to the orthogonal case, and is proved in exactly the same way (using Theorem 15.4[5] in place of Theorem 15.3[5]):

Theorem 3.1.3.
1. *Let $\rho : R \to M_{2d}(A)$ be a symplectic representation. Then $\mathrm{Tr}(\rho)$ is a symplectic pseudocharacter of dimension $2d$.*
2. *Suppose A is a field of characteristic 0, and let $\rho : R \to M_{2d}(A)$ be a symplectic representation. Then $\ker(\mathrm{Tr}(\rho)) = \ker(\rho^{ss})$.*
3. *Suppose A is an algebraically closed field of characteristic 0. Let $T : R \to A$ be a symplectic pseudocharacter of dimension $2d$. Then there is a semisimple symplectic representation $\rho : R \to M_{2d}(R, A)$ such that $\mathrm{Tr}(\rho) = T$, unique up to symplectic conjugation.*
4. *Suppose R is finitely generated as an A-algebra. Then there is a finite subset $S \subset R$ such that for any $\mathbb{Z}[1/(2d)!]$-algebra A, if $T : R \to A$ is a symplectic pseudocharacter of dimension $2d$, then T is determined by its values on A.*
5. *If R and A are taken to be topological, then the above statements hold in topological/continuous form.*

3.1.2 (General) Orthogonal Representations of Groups

Recall that the general orthogonal group over A of dimension d is defined by

$$GO_d(A) := \{A \in M_d(A) | AA^t = \lambda I \text{ for some } \lambda \in A^*\}$$

Here A^* denotes the unit group of A.

A *GO_d-pseudocharacter of G over A* is a pair (T, λ), consisting of a set map $T : G \to A$ and a group homomorphism $\lambda : G \to A^*$, such that:

- $T(1) = d$
- For all $g_1, g_2 \in G$, $T(g_1 g_2) = T(g_2 g_1)$
- For all $g \in G$, $T(g) = \lambda(g) T(g^{-1})$
- For all integers $0 \leq k \leq (d+1)/2$ and for all $g_1, \ldots, g_{d+1} \in G$, T satisfies the relation $F_{k,d+1}^{\lambda}(T, g_1, \ldots, g_{d+1})$, where $F_{k,d+1}^{\lambda}$ is the same as $F_{k,d+1}$ except that we replace any inverted element h^{-1} with $\lambda(h)h^{-1}$ and then distribute the scalars $\lambda(h)$ outside of any application of T.

Note that T is always a GL_d-pseudocharacter, since $F^{\lambda}_{0,d+1}$ is the nontrivial GL_d-pseudocharacter relation.

The kernel of a GO_d-pseudocharacter (T, λ) is defined to be the usual kernel of T, i.e.,

$$\ker(T, \lambda) := \ker(T) = \{g \in G | \text{for all } h \in G,\, T(gh) = T(g)\}$$

Then $\ker(T, \lambda) = \ker(T) \cap \ker(\lambda)$. Indeed, if $g \in \ker(T)$, then $g^{-1} \in \ker(T)$ as well, so $T(g) = T(gg^{-1}) = T(g^{-1})$; hence by the third pseudocharacter relation, $g \in \ker(\lambda)$. As usual, the kernel is a normal subgroup of G, and we have an induced GO_d-pseudocharacter on $G/\ker(T, \lambda)$.

Given a representation $\rho : G \to GO_d(A)$, define a set map $\lambda(\rho) : G \to A^*$ by letting $\lambda(g)$ be the scalar such that $\rho(g)\rho(g)^t = \lambda(g)I$. Easily $\lambda(\rho)$ is a group homomorphism.

Then we have Theorem 2.1.2 above.

Proof. Fix a group homomorphism $\lambda : G \to A^*$. Let R be the group algebra $A[G]$, with involution defined by $(g)^* = \lambda(g)(g^{-1})$. This involution is anti-multiplicative precisely because λ is a group homomorphism. We see that d-dimensional orthogonal representations of R correspond precisely to representations $\rho : G \to GO_d(A)$ with $\lambda(\rho) = \lambda$, and similarly for GO_d-pseudocharacters.

From these remarks, parts (1), (2), (4), and (5) follow immediately. Part (3) follows as well once we note that, because A is algebraically closed, conjugation by $GO_d(A)$ and $O_d(A)$ are the same thing. Indeed, if $M \in GO_d(A)$ satisfies $MM^t = rI$, then for either choice of $\sqrt{r}$, $(1/\sqrt{r})M$ is orthogonal and induces the same inner automorphism as M. □

By fixing $\lambda = 1$ in the above theorem, we get an identical theorem for orthogonal representations $\rho : G \to O(d, R)$.

The general symplectic group GSp_{2d} and symplectic group Sp_{2d} are treated similarly.

3.1.3 Special Orthogonal Representations of Groups

Odd Dimension When d is odd, the invariants of M^r_d under simultaneous conjugation by SO_d are the same as under conjugation by O_d, since every orthogonal matrix is ± 1 times a special orthogonal matrix. Thus by a result of affine geometric invariant theory (see, e.g., Lemma 2.2.1[13] and Corollary 2.4.5[13]), the invariant ring $\mathcal{O}(SO^r_d//SO_d) = \mathcal{O}(SO^r_d//O_d)$ is the quotient of $\mathcal{O}(O^r_d//O_d)$ by its intersection with the ideal $I \lhd \mathcal{O}(O^r_d)$ generated by relations $\det(A_i) = 1$, for A_i a typical matrix. Then $\mathcal{O}(SO^r_d//SO_d)$ is also generated by traces of words in the A_i and A^t_i. Also, using Hilbert's Nullstellensatz, it is easy to see that ideal of relations between these generators is the radical of the ideal generated by the O_d relations and the relations $\det(A_i) = 1$ (as expressed in terms of $\mathrm{Tr}(A_i), \ldots, \mathrm{Tr}(A^d_i)$).

Hence we define an *odd-dimensional SO_d-pseudocharacter of G over A* to be an O_d-pseudocharacter $T : G \to A$ which additionally satisfies the relation $\det(T)(g) = 1$ for all $g \in G$, where $\det(T)(g)$ is a polynomial such that $\det(\mathrm{Tr})(B) = \det(B)$ for all matrices B. Then the usual result holds by Lafforgue's Proposition 11.7.[6]

Even Dimension When d is even, the invariant theory of SO_d is more complicated. Now $\mathcal{O}(M_d^r//SO_d)$ is generated by two "typical" invariants, whose arguments range over all semigroup words in the A_i and A_i^t: $\text{Tr}(M)$ and $\text{pl}(M_1, \ldots, M_{d/2})$.[11] Here pl is the *linearized Pfaffian*, defined as the full polarization of the function

$$\widetilde{\text{pf}}(W) := \text{pf}(W - W^t)$$

where pf is the usual Pfaffian.

The relations between these invariant generators do not appear to be known, but Rogora[10] determines them up to radical:

Lemma 3.1.4. *The ideal of relations between the generators* $\text{Tr}(M)$ *and* $\text{pl}(M_1, \ldots, M_d)$ *for* $\mathcal{O}(M_d^r//SO_d)$ *is the radical of the ideal generated by the* O_d*-relations and the relations described in Theorem 3.2.*[10]

Proof. Let R be a polynomial in the given generators which evaluates to zero as a polynomial in the matrix coefficients. Note that conjugating all inputs by an element of $O_d \setminus SO_d$ preserves the value of any generator $\text{Tr}(M)$ while negates the value of any generator $\text{pl}(M_1, \ldots, M_d)$. Thus conjugating all inputs of any monomial in R sends that monomial to either itself or its negation; we call the monomial even in the former case and odd in the latter case. Let R_e and R_o be the sums of all even and odd monomials in R, respectively. Then R_e evaluates to $-R_o$ as polynomials in the matrix coefficients. But conjugating all inputs by an element of $O_d \setminus SO_d$, we see that R_e evaluates to R_o. Hence R_e and R_o both evaluate to 0, so that they are both valid relations.

It now suffices to show that the even and odd relations are in the given ideal. If R_e is an even relation, then each of its monomials consists of traces and of pairs of linearized Pfaffians. After replacing each pair of linearized Pfaffians with a polynomial in traces using the relations described in Theorem 3.2,[10] we get a polynomial in the traces which is an O_d-invariant. Hence R_e is in the given ideal. Next, if R_o is an odd relation, then R_o^2 is an even relation, hence is in the given ideal. Then R_o is in the radical. $\square$

Then we define an *even-dimensional* SO_d*-pseudocharacter of* G *over* A to be a pair of functions $T : G \to A$, $P : G^{d/2} \to A$, such that:

- T is an O_d-pseudocharacter of G over A
- For all $g \in G$, $\det(T)(g) = 1$
- For all $g_1, \ldots, g_{d/2}, h_1, \ldots, h_{d/2}$, $P(g_1, \ldots, g_{d/2})P(h_1, \ldots, h_{d/2})$ satisfies the relation in Theorem 3.2[10] with P in place of Q and T in place of Tr.

Then we have the usual result, which can be proved using Lafforgue's result, where we use Hilbert's Nullstellensatz (which applies by Hilbert's theorem on the finite generation of the ring of invariants) to account for the fact that we have only know the invariant relations up to radical.

3.2 General Existence of Pseudocharacters

3.2.1 FFS-Algebras

Let k be a commutative ring. Let FFS be the category of finitely generated free semigroups. If I is a finite set, then we will let I^+ denote the free semigroup on I. Note that a morphism $\varphi : I^+ \to J^+$ is equivalent to an assignment of an element $\varphi(i) \in J^+$ for each $i \in I$.

We define an *FFS-algebra over* k to be a functor from FFS to the category of commutative k-algebras. We will usually drop the reference to k when it is clear from context. A *morphism of FFS-algebras* is a natural transformation between two FFS-algebras. We thus form the category of FFS-algebras.

If A is an FFS-algebra and I is a finite set, then we will let A_I denote the k-algebra which is the image of I under A, and if $\varphi : I^+ \to J^+$ is a semigroup morphism, then we will let $A_\varphi : A_I \to A_J$ denote the corresponding k-algebra homomorphism.

We can define kernels, cokernels, subobjects, quotients, and coproducts in the category of FFS-algebras by using the analogous constructions in the category of k-algebras, applying those constructions to each k-algebra in the image of an FFS-algebra.

Example 3.2.1. Let S be a semigroup. Given a finite set I, let S^I denote the semigroup of I-tuples of elements of S, and let $\mathrm{Map}(S^I, k)$ denote the k-algebra of set maps from S to k. Then we can define an FFS-algebra $\mathrm{Map}(S^-, k)$ by

$$\mathrm{Map}(S^-, k)_I := \mathrm{Map}(S^I, k)$$
$$\mathrm{Map}(S^-, k)_{(\varphi:I^+\to J^+)}(f)(s_j)_{j\in J} := f\left(\varphi(i)(s_j)_{j\in J}\right)_{i\in I}$$

Here $\varphi(i)(s_j)_{j\in J}$ means the element of S^J which results when each letter j in $\varphi(i)$ is replaced by s_j. We will denote this FFS-algebra by $\mathrm{Map}(S^-, k)$.

When S and k are topological, we can replace $\mathrm{Map}(S^I, k)$ with the smaller k-algebra $C(S^I, k)$ of continuous maps (where S^I has the product topology). We will denote the resulting FFS-algebra by $C(S^-, k)$.

Example 3.2.2. Let $k[\{x_i | i \in I\}]$ denote the polynomial ring obtained by adjoining variables indexed by I. Then we can define an FFS-algebra $k[x^-]$ by

$$k[x^-]_I := k[\{x_i | i \in I\}]$$
$$k[x^-]_{(\varphi:I^+\to J^+)}(p)(q_j)_{j\in J} := p[x_i \mapsto \varphi(i)(q_j)_{j\in J}]$$

Here $\varphi(i)(q_j)_{j\in J}$ means the polynomial in $k[x^-]_J$ which results when each letter j in $\varphi(i)$ is replaced by q_j, and $x_k \mapsto r$ indicates variable substitution.

Example 3.2.3. Assume k is a field. Let S be an algebraic semigroup defined over k, and let G be a group which acts on S as a group of automorphisms. For any finite set I, we have an action of G on the variety S^I by simultaneous action on the copies of S. Then we can define an FFS-algebra $\mathcal{O}(S^-)$ by

$$\mathcal{O}(S^-)_I := \mathcal{O}(S^I)$$

together with the morphisms given in Example 3.2.1.

Now let $S^I//G := \operatorname{Spec}(k[S^I]^G)$ denote the geometric quotient of S^I by G. Then we have a subalgebra $\mathcal{O}(S^-//G)$ of $\mathcal{O}(S^-)$ defined by

$$\mathcal{O}(S^-//G)_I := \mathcal{O}(S^-//G)$$

Example 3.2.4. Assume k is a topological field. Let H be a reductive algebraic group, and let Γ be a topological group. Then letting H^0 denote the identity component of H, conjugation induces an action of H^0 on H. Then the FFS-algebra morphisms

$$\Xi : \mathcal{O}(H^-//H^0) \to C(\Gamma^-, k)$$

are in bijection with the sequences of morphisms (Ξ_n) used in Proposition 11.7.[6] We will call such morphisms *H-pseudocharacters of* Γ. By Proposition 11.7,[6] when k is an algebraically closed field of characteristic 0, H-pseudocharacters of Γ are in bijection with H^0-conjugacy classes of continuous semisimple representations $\sigma : \Gamma \to H(k)$ such that the Zariski closure of $\sigma(\Gamma)$ is reductive.

3.2.2 Finitely Generated FFS-Algebras

Let A be an FFS-algebra. Given a subset $\Sigma \subset \sqcup_I A_I$, the *span* $\operatorname{span}_A(\Sigma)$ *of* Σ *in* A is defined to be the minimum sub-FFS-algebra of A containing each element of Σ. We define an FFS-algebra to be *finitely generated* if it equals the span of some finite set.

There is another way to characterize finite generation, in terms of free FFS-algebras. Let m be a nonnegative integer, and let $\mathbf{m} := \{1, 2, \ldots, m\}$ denote the typical set of m elements. Then the *free FFS-algebra of degree* m, denoted $F(m)$, is defined by

$$F(m)_I := k[\{x_\varphi | \varphi \in \operatorname{Hom}(\mathbf{m}^+, I^+)\}]$$
$$F(m)_\psi := (x_\varphi \mapsto x_{\psi\circ\varphi})$$

If A is an FFS-algebra and $a \in A_{\mathbf{m}}$, then it is easy to see that $x_{id_{\mathbf{m}}} \mapsto a$ extends to a unique map of FFS-algebras $F(m)_I \to A$, and its image is precisely $\operatorname{span}_A(a)$. Thus:

Proposition 3.2.5. *An FFS-algebra A is finitely generated iff it admits a surjection $\bigotimes_i F(m_i) \twoheadrightarrow A$ for some finite sequence of integers (m_i), where the tensor product is taken over k.*

3.2.3 Finite Generation of Invariants

Theorem 3.2.6. *Assume k is a field of characteristic 0. Let G be a reductive group which acts as a group of automorphisms on the algebraic monoid $M_d(k)$. Then the FFS-algebra $\mathcal{O}(M_d^-//G)$ is finitely generated.*

Proof. By Hilbert's theorem on the finite generation of rings of invariants, for every finite set I, $\mathcal{O}(M_d^I//G)$ is finitely generated as a k-algebra. Let Ω be a finite set of multihomogeneous k-algebra generators for $\mathcal{O}(M_d^{d^2}//G)$. Then by Theorem 11.1.1.1,[14] for all n, $\mathcal{O}(M_d^n//G)$ consists of polarizations of elements of $\mathcal{O}(M_d^{d^2}//G)$, from which one can see that $\mathcal{O}(M_d^n//G)$ is generated by polarizations of elements of Ω. In other words, $\mathcal{O}(M_d^{\mathbb{N}}//G)$ is generated by polarizations of elements of Ω.

Now easily any polarization of a multihomogeneous function h can be obtained as a further polarization of any full polarization of h (up to a scalar multiple): insert a sequence of redundant polarizations $(y(\partial/\partial x))(x(\partial/\partial y))$ to the right of your polarization, then rearrange using commutativity so that all the right-hand parts of the redundant polarizations are on the far right, hence performing a full polarization first. So, letting Σ be a finite set containing one full polarization of each element of Ω, $\mathcal{O}(M_d^{\mathbb{N}}//G)$ is also generated by Σ under polarization.

Now let $f \in \mathcal{O}(M_d^n//G)$ for some n. We claim that f is in the $\mathbf{n}$-part of the FFS-algebra span of Σ. By the above paragraph, there are elements $g_1, \ldots, g_r$ in Σ with polarizations $g_1^1, \ldots, g_1^{i_1}, \ldots, g_r^1, \ldots, g_r^{i_r}$ such that f is the k-algebra generated by the g_j^l. Since any polarization of a full polarization results from vector variable substitutions in the full polarization, we see that each g_j^l is in the FFS-algebra span of Σ. Using the natural embeddings $\mathcal{O}(M_d^s//G) \subset \mathcal{O}(M_d^t//G)$ whenever $s < t$, which correspond to the natural embedding $\mathbf{s}^+ \subset \mathbf{t}^+$, we can assume that all g_j^l are in $\mathcal{O}(M_d^N//G)$ for some N. Then the image of f under the embedding $\mathcal{O}(M_d^n//G) \subset \mathcal{O}(M_d^N//G)$ lies in the k-algebra generated by the g_j^l. Mapping this image of f back to $\mathcal{O}(M_d^n//G)$ using the map $\mathcal{O}(M_d^-//G)_\varphi$ for some $\varphi : \mathbf{N}^+ \to \mathbf{n}^+$ which is the identity on $\mathbf{n}^+$, we thus find that f is in the FFS-algebra span of Σ. □

Remark 3.2.7. The proof shows that finite generation would still hold if instead of using FFS-algebras, we use a weaker structure consisting of a functor from the category of finite sets to the category of k-algebras, with finite generation defined in the obvious way.

Corollary 3.2.8. *In the situation of the above theorem, let S be an algebraic sub-semigroup of $M_d(k)$ which is stable under the action of G. Then $\mathcal{O}(S^-//G)$ is finitely generated.*

Proof. Let $\mathfrak{a} \subset k[M_d]$ be the ideal defining S. Then by a result of affine geometric invariant theory (see, e.g., Lemma 2.2.1[13] and Corollary 2.4.5[13]), for all finite sets I,

$$k[S^I]^G = \frac{k[M_d^I]^G}{\mathfrak{a}^I \cap k[M_d^I]^G}$$

Hence we can exhibit $\mathcal{O}(S^-//G)$ as a quotient of $\mathcal{O}(M_d^-//G)$, and obviously any quotient of a finitely generated FFS-algebra is finitely generated. □

In particular, pseudocharacters of general reductive groups (under conjugation by their identity component) can always be defined in terms of a finite number of functions $G^I \to k$.

3.3 Element-conjugacy and Conjugacy of Representations

3.3.1 General Principles

Let G be a linear algebraic group for which pseudocharacters have been defined. Then as discussed in Section 2.3, if these pseudocharacters consist of one-argument functions only, then semisimple element-conjugant representations into G are globally conjugate.

When G is a connected reductive group, so that Lafforgue's result applies, we can state this result more formally using a structure similar to FFS-algebras. Specifically, let FFG denote the category of free finitely generated groups, and define FFG-algebras analogously to the FFS-algebras of the previous section. Then we can make $\mathcal{O}(G^-//G)$ and $C(\Gamma^-, k)$ into FFG-algebras in a natural way for any group Γ. From Proposition 11.7,[6] we see that G-conjugacy classes of semisimple representations of Γ are in bijection with FFG-algebra morphisms $(G^{-}//G) \to C(\Gamma^{-}, k)$. If $\mathcal{O}(G^{-}//G)$ is generated (in a similar sense to FFS-algebra generation) by its degree-1 part $\mathcal{O}(G//G)$, then G has pseudocharacters consisting of one-argument functions only, hence element-conjugacy implies global conjugacy. We conjecture that the converse holds as well, although we do not yet see how to prove such a statement:

Conjecture 3.3.1. *Let G be a connected reductive group. Then element-conjugacy implies conjugacy for all semisimple representations $\Gamma \to G$ (including all groups Γ) iff $\mathcal{O}(G^{-}//G)$ is generated by $\mathcal{O}(G//G)$.*

3.3.2 Element-conjugacy and Conjugacy for $SU(3) \rtimes \langle\tau\rangle$

Let τ denote the conjugation automorphism of $M_3(\mathbb{C})$ as an R-algebra, mapping A to $\overline{A}$. Let $G = SU(3) \rtimes \langle\tau\rangle$, where the action of τ on $SU(3)$ is the restriction of its action on $M_3(\mathbb{C})$. Then we have an action of G on $M_3(\mathbb{C})$, by

$$(A, 1).B = ABA^{-1}$$
$$(A, \tau).B = A\overline{B}A^{-1}$$

Let S be the free *-algebra in n variables, i.e., $S = \mathbb{C}[a_1, \ldots, a_n, b_1, \ldots, b_n]$ with $\alpha^* = \overline{\alpha}$ for $\alpha \in \mathbb{C}$, and $a_i^* = b_i$. We can make $M_3(\mathbb{C})$ into a *-algebra by defining $\alpha^* = \overline{\alpha}$ for $\alpha \in \mathbb{C}$, and $A^* = \overline{A^t}$. We then refer to *-algebra representations $S \to M_3(\mathbb{C})$ as unitary representations. By Theorem 16.5,[5] two semisimple unitary representations $S \to M_3(\mathbb{C})$ are equivalent under conjugation by $U(3)$ iff the points

$$\rho_i' = (\rho_i(a_1), \rho_i(a_1^*), \ldots, \rho_i(a_n), \rho_i(a_n^*)) \in M_3^{2n}(\mathbb{C})$$

are equivalent under simultaneous conjugation by $GL_3(\mathbb{C})$. By a standard result of representation theory over $\mathbb{C}$, this is iff for all $s \in S$, $\mathrm{Tr}(\rho_1(s)) = \mathrm{Tr}(\rho_2(s))$.

Thus two unitary representations of S are equivalent under conjugation by $U(3)$ iff they have the same trace. Since every element of $U(3)$ is a scalar multiple of an element of $SU(3)$, the same statement holds for $SU(3)$. This result extends immediately to the case that S is a finitely generated *-algebra. By the same reasoning as in p. 286,[4] the result extends to general *-algebras S.

We now adapt this condition to the action of G instead of $SU(3)$.

Lemma 3.3.2. *Let S be a *-algebra over $\mathbb{C}$, with * acting as complex conjugation on $\mathbb{C}$. Two semisimple unitary representations $\rho_1, \rho_2 : S \to M_3(\mathbb{C})$ are equivalent under the action of G on $M_3(\mathbb{C})$ iff for all $s \in S$,*

$$\mathrm{Re}(\mathrm{Tr}(\rho_1(s))) = \mathrm{Re}(\mathrm{Tr}(\rho_2(s)))$$
$$\mathrm{Im}(\mathrm{Tr}(\rho_1(s)))^2 = \mathrm{Im}(\mathrm{Tr}(\rho_2(s)))^2$$

Proof. If ρ_1 and ρ_2 are equivalent under the action of G, then ρ_1 is equivalent to either ρ_2 or $\overline{\rho_2}$ under the action of $SU(3)$. Hence by the above remarks, for all $s \in S$, $\mathrm{Tr}(\rho_1(s)) = \mathrm{Tr}(\rho_1(s))$ or $\overline{\mathrm{Tr}(\rho_1(s))}$, proving the claim.

Conversely, suppose the given trace conditions hold. If $\mathrm{Im}(\mathrm{Tr}(\rho_1(s))) = 0$ for all $s \in S$, then we are done by the above remarks on $SU(3)$, so assume there is an $s_0 \in S$ such that $\mathrm{Im}(\mathrm{Tr}(\rho_1(s_0))) \neq 0$. Because

$$\begin{aligned}\mathrm{Im}(\mathrm{Tr}(\rho_1(s+t)))^2 &= (\mathrm{Im}(\mathrm{Tr}(\rho_1(s))) + \mathrm{Im}(\mathrm{Tr}(\rho_1(t))))^2 \\ &= \mathrm{Im}(\mathrm{Tr}(\rho_1(s)))^2 + 2\mathrm{Im}(\mathrm{Tr}(\rho_1(s)))\mathrm{Im}(\mathrm{Tr}(\rho_1(t))) + \mathrm{Im}(\mathrm{Tr}(\rho_1(t)))^2,\end{aligned}$$

the second trace condition implies that

$$\mathrm{Im}(\mathrm{Tr}(\rho_1(s)))\mathrm{Im}(\mathrm{Tr}(\rho_1(t))) = \mathrm{Im}(\mathrm{Tr}(\rho_2(s)))\mathrm{Im}(\mathrm{Tr}(\rho_2(t)))$$

for all $s, t \in S$. Obviously $\mathrm{Im}(\mathrm{Tr}(\rho_1(s))) = \pm\mathrm{Im}(\mathrm{Tr}(\rho_2(s)))$ for all $s \in S$. If $\mathrm{Im}(\mathrm{Tr}(\rho_1(s_0))) = \mathrm{Im}(\mathrm{Tr}(\rho_2(s_0)))$, then the above relations imply that $\mathrm{Im}(\mathrm{Tr}(\rho_1(s))) = \mathrm{Im}(\mathrm{Tr}(\rho_2(s)))$ for all $s \in S$, so the representations are equivalent under conjugation by $SU(3)$. If instead $\mathrm{Im}(\mathrm{Tr}(\rho_1(s_0))) = -\mathrm{Im}(\mathrm{Tr}(\rho_2(s_0)))$, then $\mathrm{Im}(\mathrm{Tr}(\rho_1(s))) = -\mathrm{Im}(\mathrm{Tr}(\rho_2(s)))$ for all $s \in S$, so ρ_1 and $\overline{\rho_2}$ are equivalent under conjugation by $SU(3)$. Hence in either case, ρ_1 and ρ_2 are equivalent under the action of G. □

In particular, we can apply this lemma to the case that S is the group algebra $\mathbb{C}[\Gamma]$ for some group Γ, with * defined by $(\gamma)^* = (\gamma^{-1})$. Then semisimple unitary representations of S are in bijection with semisimple unitary representations $\Gamma \to U(3)$. If two semisimple unitary representations of Γ are element-conjugate under G, i.e., $\rho_1(g)$ and $\rho_2(\Gamma)$ are in the same orbit under G for all $\gamma \in \Gamma$, then the trace conditions in the above theorem hold for all $x \in \mathbb{C}[\Gamma]$, so ρ_1 and ρ_2 are equivalent under the action of G. Hence:

Lemma 3.3.3. *Let Γ be a group, and let $\rho_1, \rho_2 : \Gamma \to U(3)$ be semisimple representations. Then ρ_1 and ρ_2 are equivalent under the action of G on $U(3)$ (induced by the action of G on $M_3(\mathbb{C})$) iff for all $\gamma \in \Gamma$, $\rho_1(\gamma)$ and $\rho_2(\gamma)$ are in the same orbit under G.*

We can now generalize Lemma 2.7[7] to arbitrary groups Γ:

Lemma 3.3.4. *Let Γ be a group, and let $\rho_1, \rho_2 : \Gamma \to G$ be homomorphisms. Assume:*

- *$\rho_1|_{\rho_1^{-1}(SU(3))}$ and $\rho_2|_{\rho_2^{-1}(SU(3))}$ are semisimple representations.*
- *For all $\gamma \in \Gamma$, $\rho_1(\gamma)$ and $\rho_2(\gamma)$ are conjugate in G.*

Then ρ_1 and ρ_2 are conjugate in G.

Proof. The compositions of ρ_1 and ρ_2 with $G \twoheadrightarrow \langle \tau \rangle$ are identical by the second assumption, so we can let

$$\Gamma' := \rho_1^{-1}(SU(3)) = \rho_2^{-1}(SU(3))$$

Let ρ_1' and ρ_2' be the restrictions of ρ_1 and ρ_2 to Γ', respectively. Then for all $\gamma \in \Gamma'$, $\rho_1'(\gamma)$ and $\rho_2'(\gamma)$ are in the same orbit of $SU(3)$ under the action of G. Hence by the previous lemma, ρ_1' and ρ_2' are equivalent under the action of G. Thus conjugating ρ_2, WLOG $\rho_1' = \rho_2'$.

From here, we can follow the same proof as in Lemma 2.7,[7] starting at the first full paragraph of pg. 264. □

3.3.3 Element-conjugacy and Conjugacy for G_2

Let $G_2(\mathbb{R})$ denote the compact form of G_2, and let $G_2(\mathbb{C})$ denote its complex form.

Theorem 3.3.5. *Let Γ be a compact Hausdorff topological group. Suppose $\rho_1, \rho_2 : \Gamma \to G_2(\mathbb{R})$ are two continuous semisimple representations of Γ such that for all $\gamma \in \Gamma$, $\rho_1(\gamma)$ and $\rho_2(\gamma)$ are conjugate in $G_2(\mathbb{R})$. Then ρ_1 and ρ_2 are globally conjugate in $G_2(\mathbb{R})$.*

Proof. We closely follow the proof of Proposition 2.8,[7] which handles the case when Γ is finite. Compose the ρ_i with the irreducible representation $\psi : G_2(\mathbb{R}) \to SO(7, \mathbb{R})$, where the first map is given by the action of G_2 on traceless real Cayley numbers. Then $\psi \circ \rho_1$ and $\psi \circ \rho_2$ are semisimple representations of Γ with image in $SO(7, \mathbb{R})$ such that for all $\gamma \in \Gamma$, $(\psi \circ \rho_1)(\gamma)$ and $(\psi \circ \rho_2)(\gamma)$ are conjugate in $SO(7, \mathbb{R})$. By Corollary 2.6,[9] these two representations are globally conjugate in $SO(7, \mathbb{R})$. That is, there exists $g \in SO(7, \mathbb{R})$ such that for all $\gamma \in \Gamma$,

$$\psi(\rho_1(\gamma)) = g\psi(\rho_2(\gamma))g^{-1}.$$

Thus

$$\psi(\rho_1(\Gamma)) \subset \psi(G_2(\mathbb{R})) \cap g\psi(G_2(\mathbb{R}))g^{-1}.$$

This intersection of 14-dimensional subgroups of a 21-dimensional group must have dimension ≥ 7. Its identity component is therefore a compact connected Lie group of reductive rank ≤ 2 and dimension ≥ 7 which admits an embedding in $G_2(\mathbb{R})$. As in the original proof, the identity component is isomorphic to either $G_2(\mathbb{R})$ or $SU(3)$. In the $G_2(\mathbb{R})$ case, the proof is the same as in Proposition 2.8,[7] while in the $SU(3)$ case, it is the same except that we replace Lemma 2.7[7] with Lemma 3.3.4 above. ☐

Corollary 3.3.6. *Let Γ be a compact Hausdorff topological group. Suppose $\rho_1, \rho_2 : \Gamma \to G_2(\mathbb{C})$ are two continuous semisimple representations of Γ such that for all $\gamma \in \Gamma$, $\rho_1(\gamma)$ and $\rho_2(\gamma)$ are conjugate in $G_2(\mathbb{C})$. Then ρ_1 and ρ_2 are globally conjugate in $G_2(\mathbb{C})$.*

Proof. Apply the forwards direction of Proposition 1.7,[7] whose proof still works when $\rho_1(\Gamma)$ and $\rho_2(\Gamma)$ are compact instead of finite. ☐

3.3.4 Element-conjugacy and Conjugacy for SO_{2d}

Let k be an algebraically closed field of characteristic 0, and let $d \geq 2$ be an integer. We want to characterize all pairs of representations $\rho, \rho' : G \to SO_{2d}(k)$ such that $\rho(g)$ is SO_{2d}-conjugate to $\rho'(g)$ for all $g \in G$, but ρ is not SO_{2d}-conjugate to ρ'. Let pl denote the linearized antisymmetrized Pfaffian.[11] Our result is as follows.

Proposition 3.3.7. *Let G be a group, and let $\rho : G \to SO_{2d}(k)$ be a representation. Then there is a representation which is element-conjugate but not globally conjugate to ρ in $SO_{2d}(k)$ iff:*

- *For all $g \in G$,* $\det(\rho(g) - \rho(g)^t) = 0$
- *There exist $g_1, \ldots, g_d \in G$ such that* $\mathrm{pl}(\rho(g_1), \ldots, \rho(g_d)) \neq 0$.

In this situation, there is a unique such ρ' up to equivalence under $SO_{2d}(k)$, and it is given by

$$\rho'(g) = X\rho(g)X^{-1}$$

for any $X \in O_{2d}(k) \setminus SO_{2d}(k)$.

Proof. **Uniqueness:** Let ρ' be element-conjugate but not globally conjugate to ρ in SO_{2d}. Then ρ and ρ' are element-conjugate in O_{2d}, hence globally conjugate in O_{2d}. Thus there is an $X \in O_{2d}(k)$ such that $\rho' = X\rho X^{-1}$. Since ρ and ρ' are not globally conjugate, conjugation by X must induce an outer automorphism of SO_{2d}. Since $SO_{2d}(k)$ has index 2 in $O_{2d}(k)$, easily any other choice of X gives a representation which is conjugate to ρ' in $SO_{2d}(k)$.

Existence, ($\Longrightarrow$): Let $\rho' = X\rho X^{-1}$ as above. Let pl denote the linearized antisymmetrized Pfaffian, which is an odd d-ary invariant of $\mathcal{O}(SO_{2d}^{\mathbb{N}}//SO_{2d})$. Here odd means that

$$\mathrm{pl}(\rho(g_1), \ldots, \rho(g_d)) = -\mathrm{pl}(\rho'(g_1), \ldots, \rho'(g_d))$$

for all $g_1, \ldots, g_d \in G$. But since ρ and ρ' are element-conjugate, $\rho|_{\langle g \rangle}$ is conjugate to $\rho'|_{\langle g \rangle}$ in SO_{2d} for each $g \in G$, so

$$\mathrm{pl}(\rho(g^{m_1}), \ldots, \rho(g^{m_d})) = \mathrm{pl}(\rho'(g^{m_1}), \ldots, \rho(g^{m_d}))$$

for all $g \in G$ and $m_1, \ldots, m_d \in \mathbb{Z}$. Hence $\mathrm{pl}(\rho(g^{m_1}), \ldots, \rho(g^{m_d})) = 0$. In particular, $\tilde{\mathrm{pf}}(\rho(g)) = \mathrm{pl}(\rho(g), \ldots, \rho(g)) = 0$ for all $g \in G$. Hence

$$\det(\rho(g) - \rho(g)^t) = \mathrm{pf}(\rho(g) - \rho(g)^t)^2 = \tilde{\mathrm{pf}}(\rho(g))^2 = 0.$$

Existence, ($\Longleftarrow$): Let $\rho'(g) = X\rho(g)X^{-1}$ where X is as in the proof of uniqueness. Then by assumption, there exist $g_1, \ldots, g_d$ such that

$$\mathrm{pl}(\rho(g_1), \ldots, \rho(g_d)) \neq -\mathrm{pl}(\rho(g_1), \ldots, \rho(g_d)) = \mathrm{pl}(\rho'(g_1), \ldots, \rho'(g_d)),$$

so ρ and ρ' are not globally conjugate.

Now fix $g \in G$. To show that $\rho|_{\langle g \rangle}$ and $\rho'_{\langle g \rangle}$ are conjugate in SO_{2d}, it suffices to show that they have the same SO_{2d}-pseudocharacters. They have the same traces because ρ and ρ' are conjugate in O_{2d}. To show that they have the same linearized Pfaffians, we must show

$$\mathrm{pl}(\rho(g^{m_1}), \ldots, \rho(g^{m_d})) = 0$$

for all $m_1, \ldots, m_d \in \mathbb{Z}$, since the corresponding Pfaffian for ρ' is the negative of that for ρ. By definition, $\mathrm{pl}(\rho(g^{m_1}), \ldots, \rho(g^{m_d}))$ is the t-multilinear term in

$$\tilde{\mathrm{pf}}(t_1\rho(g^{m_1}) + \cdots + t_d\rho(g^{m_d})) = \mathrm{pf}(t_1(\rho(g^{m_1}) - \rho(g^{m_1})^t) + \cdots + t_d(\rho(g^{m_d}) - \rho(g^{m_d})^t).$$

But $\rho(g) - \rho(g)^t = \rho(g) - \rho(g)^{-1}$ divides $\rho(g)^{m_i} - \rho(g)^{-m_i} = \rho(g^{m_i}) - \rho(g^{m_i})^t$ for all i, so the assumption $\det(\rho(g) - \rho(g)^t) = 0$ implies that

$$\det(t_1(\rho(g^{m_1}) - \rho(g^{m_1})^t) + \cdots + t_d(\rho(g^{m_d}) - \rho(g^{m_d})^t) = 0.$$

Hence taking the square root, the Pfaffian is zero as well for all values of $t_1, \ldots, t_d$. Thus $\mathrm{pl}(\rho(g^{m_1}), \ldots, \rho(g^{m_d})) = 0$, proving the claim. □

3.3.5 A Finite Abelian Counterexample to Element-conjugacy Implying Global Conjugacy

Let $\Gamma = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, with generators $(1, 0)$ and $(0, 1)$. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SO_2(\mathbb{C})$$

Define a homomorphism $\rho_6 : \Gamma \to SO_6(\mathbb{C})$ by:

$$\begin{aligned} \rho_6(1, 0) &= A \oplus A \oplus I \\ \rho_6(0, 1) &= I \oplus A \oplus A \end{aligned}$$

Then one can check that $\det(\rho_6(\gamma) - \rho_6(\gamma)^t) = 0$ for all $\gamma \in \Gamma$, while $\mathrm{pl}(\rho_6(1, 0), \rho_6(0, 1), \rho_6(0, 1)) = 16$. Hence ρ_6 is a counterexample to element-conjugacy implying conjugacy for $SO_6(\mathbb{C})$.

More generally, we have:

Proposition 3.3.8. *Let Γ and A be as above. For any $d \geq 3$, the homomorphism $\rho_{2d} : \Gamma \to SO_{2d}(\mathbb{C})$ defined by*

$$\begin{aligned} \rho_{2d}(1, 0) &= A \oplus A \oplus I \oplus \bigoplus_{i=4}^{d} A \\ \rho_{2d}(0, 1) &= I \oplus A \oplus A \oplus \bigoplus_{i=4}^{d} A \end{aligned}$$

satisfies $\det(\rho_{2d}(\gamma) - \rho_{2d}(\gamma)^t) = 0$ *for all* $\gamma \in \Gamma$ *and* $\mathrm{pl}(\rho_{2d}(1, 0), \rho_{2d}(0, 1), \ldots, \rho_{2d}(0, 1)) \neq 0$. *Hence ρ_{2d} gives a counterexample to element-conjugacy implying global conjugacy.*

Proof. We have

$$\begin{aligned} \det\left(\left(\bigoplus_{i=1}^{d} B^{(i)}\right) - \left(\bigoplus_{i=1}^{d} B^{(i)}\right)^t\right) &= \det\left(\bigoplus_{i=1}^{d} (B^{(i)} - (B^{(i)})^t)\right) \\ &= \prod_{i=1}^{d} \det(B^{(i)} - (B^{(i)})^t) \end{aligned}$$

Hence to show $\det(\rho(\gamma) - \rho(\gamma)^t) = 0$, it suffices to prove that some 2×2 diagonal block $B^{(i)}$ of $\rho(\gamma)$ satisfies $\det(B^{(i)} - (B^{(i)})^t) = 0$. But one can check that for all $\gamma \in \Gamma$, one of the first three 2×2 diagonal blocks is a symmetric matrix.

Next, recall that $\mathrm{pl}(B_1, \ldots, B_d)$ is defined to be the coefficient of $t_1 \cdots t_d$ in $\mathrm{pf}(t_1(B_1 - B_1^t) + \cdots + t_d(B_d - B_d^t))$. Letting each $B_j = \bigoplus_{i=1}^{d} B_j^{(i)}$ for some 2×2 matrices $B_j^{(i)}$, we have

$$\mathrm{pf}(t_1(B_1 - B_1^t) + \cdots + t_d(B_d - B_d^t)) = \prod_{i=1}^{d} \mathrm{pf}(t_1(B_1^{(i)} - (B_1^{(i)})^t) + \cdots + t_1(B_d^{(i)} - (B_d^{(i)})^t)$$

Now pf is a linear function of 2×2 antisymmetric matrices, so this equals

$$\prod_{i=1}^{d} \sum_{j=1}^{d} t_j \mathrm{pf}(B_j^{(i)} - (B_j^{(i)})^t)$$

Taking the coefficient of $t_1 \dots t_d$ in this formula, we find that

$$\mathrm{pl}(B_1, \dots, B_d) = \sum_{\sigma \in S_d} \prod_{i=1}^{d} \mathrm{pf}(B_{\sigma(i)}^{(i)} - (B_{\sigma(i)}^{(i)})^t)$$

Finally, note that $\mathrm{pf}(A - A^t) = 2$ and $\mathrm{pf}(I - I^t) = 0$. Thus

$$\mathrm{pl}(C_1 := \rho_{2d}(1,0), C_2 := \rho_{2d}(0,1), \dots, C_d := \rho_{2d}(0,1))$$

will be nonzero so long as for some $\sigma \in S_d$, for all i, $C_{\sigma(i)}^{(i)} = A$. Taking σ to be the identity permutation works. □

Corollary 3.3.9. *For all $d \geq 3$ and all odd primes p such that $\left(\frac{-1}{p}\right) = 1$, there is a continuous semisimple representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to SO_{2d}(\mathbb{C})$ which is a counterexample to element-conjugacy implying global conjugacy.*

Proof. In light of the above proposition and Maschke's theorem, it suffices to prove that there is an extension K of $\mathbb{Q}_p$ with $\mathrm{Gal}(K/\mathbb{Q}_p) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. By assumption, $\mu_4 \subset \mathbb{Q}_p$, so Kummer theory tells us that this will be true iff $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^4$ has a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. By the theory of local fields (see, e.g., Proposition 6.8[15]), we have

$$\left|\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^4\right| = 16$$
$$\left|\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2\right| = 4$$

Hence $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^4 \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, so adjoining all fourth roots to $\mathbb{Q}_p$ gives the desired extension K. □

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