On Decoding Cohen-Haeupler-Schulman Tree Codes

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joint with Anand Kumar Narayanan[2]

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SODA 2020

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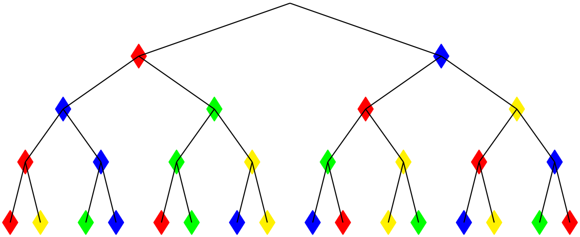
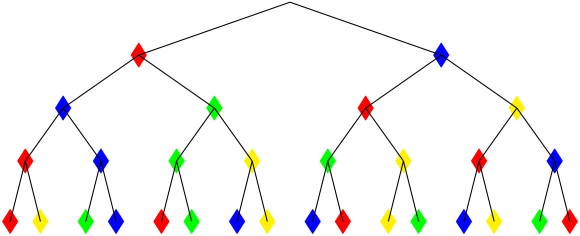
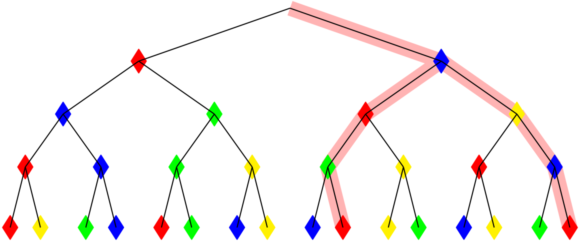




Goal: different branches have very different color sequences (after they diverge).



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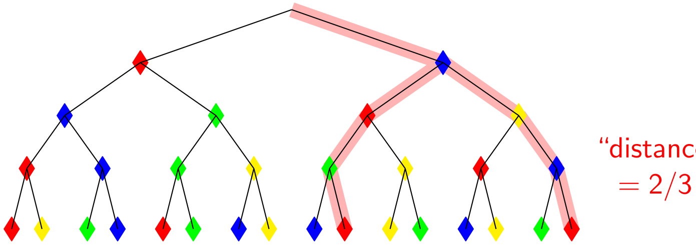

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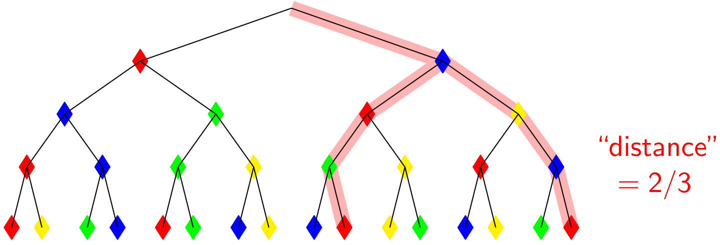
Tree Codes



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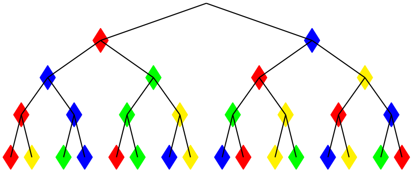
Parameters:

- Length (depth)
- Alphabet size (number of colors/labels)
- Distance (minimum distance between two branches)

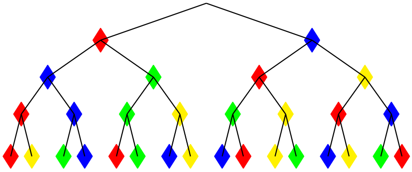


$\approx$

Online function

$\{0,1\}^{\leq n} \to \Sigma^{\leq n}$,

some alphabet Σ.



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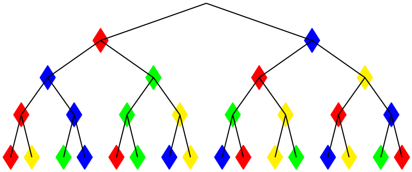

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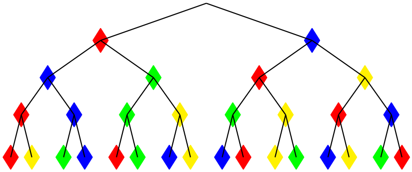

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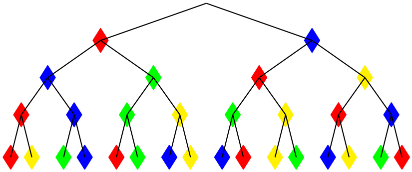
- (Schulman 90s): Add error tolerance to *interactive communication protocols*



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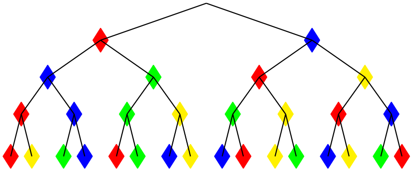

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- Explicit constructions are challenging
  - "Good" tree codes exist, but no poly-time construction is known!

(Binary) tree codes $TC : \{0,1\}^{\leq n} \to \Sigma^{\leq n}$, $|\Sigma| = \text{polylog}(n)$, distance $= \frac{1}{2}$.

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$$(z_0, z_1, \ldots, z_{n-1}) \mapsto ((z_0, a_0), (z_1, a_1), \ldots, (z_{n-1}, a_{n-1}))$$

under the Newton basis transformation

$$z_i = \sum_{j=0}^{n-1} a_j \binom{i}{j}, \qquad \forall i$$

with the inversion formula

$$a_j = \sum_{i=0}^{n-1} (-1)^{j-i} \binom{j}{i} z_i, \qquad \forall j.$$

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$$p(i) = z_i,$$

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Theorem

If $z_0 \neq 0$*, then*

$$Sparsity(z_0, z_1, \ldots, z_{n-1}) + Sparsity(a_0, a_1, \ldots, a_{n-1}) \geq n + 1$$

$$\Rightarrow Sparsity((z_0, a_0), (z_1, a_1), \ldots, (z_{n-1}, a_{n-1})) \geq \frac{n}{2}.$$

Thus the CHS code has distance $1/2$.

Restated:

$$(z_0, z_1, \ldots, z_{n-1}) \mapsto ((z_0, a_0), (z_1, a_1), \ldots, (z_{n-1}, a_{n-1})),$$

$$p(x) = \sum_{j=0}^{n-1} a_j \binom{x}{j}, \qquad z_i = p(i), \ \ \forall i.$$

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If $p(0) \neq 0$, then the number of roots of $p(x)$ in $\mathbb{N}$ is less than its sparsity in the Newton basis $\{\binom{x}{j}\}_{j \geq 0}$.

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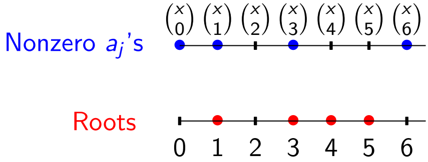

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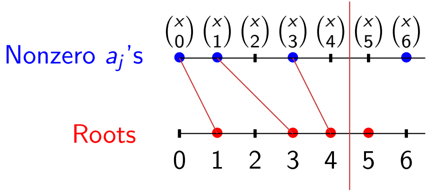

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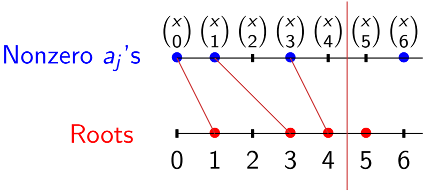

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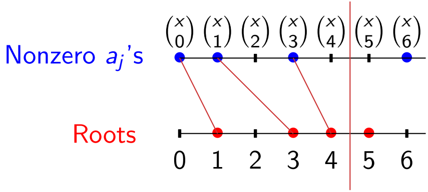

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Then $M_{IJ} \cdot \begin{pmatrix} a_{j_1} & \cdots & a_{j_k} \end{pmatrix}^T = \vec{0}$, where M_{IJ} is the I, J minor of

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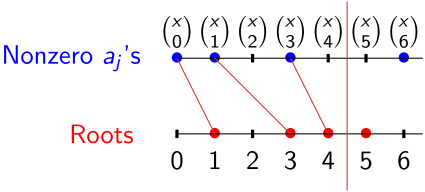

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But by the **Lindström-Gessel-Viennot Lemma**, $\det M_{IJ} \neq 0$. □

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3. Speculative framework for decoding via convex optimization

$$(z_0, z_1, \ldots, z_{n-1}) \mapsto ((z_0, a_0), (z_1, a_1), \ldots, (z_{n-1}, a_{n-1})),$$

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Given a "received word" $((\widehat{z_0}, \widehat{a_0}), (\widehat{z_1}, \widehat{a_1}), \ldots, (\widehat{z_{n-1}}, \widehat{a_{n-1}}))$ such that $(\widehat{z_i}, \widehat{a_i}) = (z_i, a_i)$ except at $< n/4$ coordinates, output z_0, in time poly(n).

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More generally, given the above and the prefix $z_0, \ldots, z_{k-1}$, if $(\widehat{z_i}, \widehat{a_i}) = (z_i, a_i)$ except at $< (n-k)/4$ coordinates $i \geq k$, output z_k.

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This yields a decoding algorithm for the binary tree codes, correcting up to $< n/4$ errors.

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This yields a decoding algorithm for the binary tree codes, correcting up to ~~$< n/4$~~ $\Omega(n^{3/4}/\sqrt{\log(n)})$ *errors.*

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1. Locate $\alpha(n)/2$ indices $i \in [0, \alpha(n))$ such that $\widehat{z_i} = z_i$, for some $\alpha(n) = \Omega(\sqrt{n \log(n)})$, using a Newton basis analog of the Sudan and Shokrollahi-Wasserman algorithms.

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3. Among the first $\alpha(n)$ coordinates, we have as many known z_i's as unknown a_j's. Interpolate the remaining unknown a_j's using the Lindström-Gessel-Viennot Lemma, recovering $z_0 = a_0$.

Decoding: Step 1

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Let $p(x) = \sum_{j=0}^{n-1} a_j \binom{x}{j}$, the correct polynomial. So $p(i) = z_i$.
Bound on # errors $\implies p(x)$ is $O(\sqrt{n/\log(n)})$-sparse, and the $\widehat{z_i}$ are the outputs of $p(x)$ except with $O(\sqrt{n/\log(n)})$ errors.

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Let $p(x) = \sum_{j=0}^{n-1} a_j \binom{x}{j}$, the correct polynomial. So $p(i) = z_i$.
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Plan: Show $b(x) + p(x)c(x)$ is sparse but also has many roots, hence is ≈ 0. (Specifically, $b(i_0) + p(i_0)c(i_0) = 0$.)

Algorithm

Let $\alpha = \alpha(n) = r\sqrt{n\log(n)}$, some $r \in \mathbb{R}$.

- Randomly choose indices $R \subset [\alpha, n)$, $|R| = n - 2\alpha + 1$.
- Find polynomials

$$b(x) = \sum_{j\in[0,\alpha)\cup R} b_j \binom{x}{j}, \qquad c(x) = \sum_{j=0}^{\alpha-1} c_j x^j$$

$$\forall i \in [0, n),\ b(i) + c(i)\widehat{z_i} = 0.$$

- Output smallest $i_0 \in [0, \alpha)$ s.t. $c(i_0) \neq 0$.

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Then $c(i_0) \neq 0 \Rightarrow \widehat{z_{i_0}} = p(i_0) = z_{i_0}$, as desired.

Decoding: Step 1

$$\alpha = O(\sqrt{n\log(n)}), \quad |R| = n - 2\alpha + 1, \quad b(x) = \sum_{j\in[0,\alpha)\cup R} b_j \binom{x}{j}, \quad c(x) = \sum_{j=0}^{\alpha-1} c_j x^j$$

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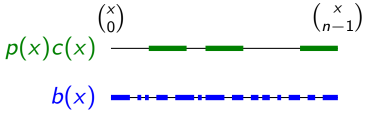

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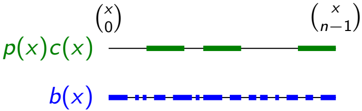

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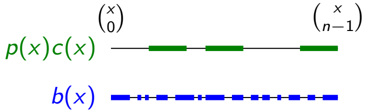

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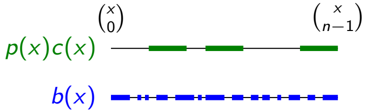

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   In the Reed-Solomon code version, these are degrees, which add.
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Treat $((\widehat{a_0}, \widehat{z_0}), (-\widehat{a_1}, -\widehat{z_1}), (\widehat{a_2}, \widehat{z_2}), \dots)$ as an errored encoding of $(a_0, -a_1, a_2, \dots)$ and apply step 1 again.

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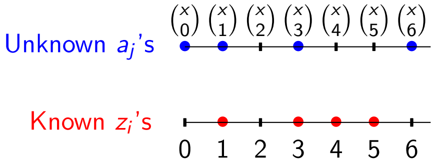

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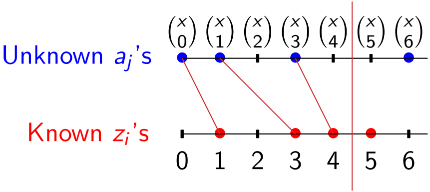

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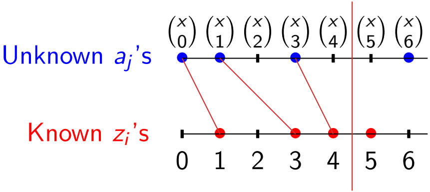

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$$M_{IJ} \cdot \begin{pmatrix} a_{j_1} \\ \vdots \\ a_{j_k} \end{pmatrix} = \begin{pmatrix} z_{i_1} = \widehat{z_{i_1}} \\ \vdots \\ z_{i_k} = \widehat{z_{i_k}} \end{pmatrix}, \quad M = \left\{ \binom{i}{j} \right\}_{0 \le i,j \le n-1}$$

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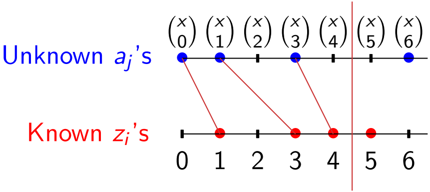

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All $j_\ell \le i_\ell \Longrightarrow \det(M_{IJ}) \ne 0$ by LGV. Solve for $a_{j_1} = a_0 = z_0$. □

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- $q = \zeta_\ell$ for $\ell > n^3$ prime
- $q = e^{2\pi\iota\theta}$ for $\theta \in \mathbf{R}$ irrational & algebraic (w/ rounding)

Speculative Framework for Decoding via Convex Optimization

Decoding problem: given $\widehat{x}$ = errored received output, solve

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Speculative Framework for Decoding via Convex Optimization

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Suffices to construct such an F with shape:

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However, this appears impossible. Need new "online RIP".

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Without online condition, Beerliová-Trubíniová and Hirt construct

$$M = \left\{ \prod_{k \neq j} \frac{\beta_i - \alpha_k}{\alpha_j - \alpha_k} \right\}_{0 \leq i,j \leq n-1}$$

over $\mathbb{F}_{2n} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$. (Interpolate $g(\alpha_j) = z_j$, then output $(g(\beta_i))_i$.)