Motivating Teleparallel Gravity

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Abstract

We present a novel motivation for teleparallel gravity. That is, we argue in favor of teleparallel gravity—known to be equivalent to general relativity—without appealing to a gauge theory for the translation group or general relativity's geometric principles.

Instead, we start with Newtonian gravity in flat spacetime and pursue a relativistic analog. Using properties of waves, we quickly derive teleparallel gravity's tetrad field, Weitzenböck connection, and torsion force. We then invoke a basic gauge principle (local Lorentz invariance) to determine the teleparallel wave equation. For completeness, we also review the well-known equivalence between teleparallel gravity and general relativity.

We hope that our approach can be useful as a teaching tool for general relativity. Our motivation requires knowledge only of special relativity, the covariant form of electromagnetism, and basic quantum mechanics; the reader does not need to be familiar with manifolds, Lagrangian field theory, or gauge theory. However, we sketch the relation to those concepts in remarks.

1 Introduction

Newtonian gravity may be described as the theory of a scalar field $\phi(t, x)$—the gravitational potential—which has the following properties:

1. For any object with (relativistic) energy E, the field grants it a potential energy term $\phi(t, x)E/c^2$.
2. The field is coupled to energy density $\rho(t, x)$ via a wave equation

$$\nabla^2\phi = -4\pi G\rho. \tag{1}$$

This theory is not compatible with special relativity, in particular because neither ϕ nor ρ are Lorentz-covariant quantities.

The goal of this paper is to pursue a relativistic analog of the above theory, as a classical field theory in flat spacetime. We will end up with the same field and equations as *teleparallel gravity*, an alternative to general relativity that is nonetheless equivalent. For a traditional introduction to teleparallel gravity, see Aldrovandi and Pereira (2013).[1]

Teleparallel gravity is typically motivated by applying gauge theory to the group of spacetime translations. While this is mathematically elegant and follows the path used to explain the other three fundamental forces, it is also rather advanced, and it requires a non-standard use of gauge theory. We will use a different, simpler argument. Besides its potential use as a teaching tool, we hope that this argument illustrates the *necessity* of teleparallel gravity as the Lorentz-invariant extension of Newtonian gravity.

1.1 Outline

An outline of our argument is as follows:

1. For any quantum system, adding the Newtonian gravitational potential to its Schrödinger equation causes the perception of gravitational time dilation. (Section 2)

2. To make time dilation Lorentz-covariant, we must replace the gravitational potential $\phi(t, x)$ with a matrix-valued field $h^a{}_\rho(t, x)$, the *tetrad field*, which changes observers' perception of both time and space. It encompasses the inertial effects of gravity, e.g., frame dragging. (Section 3)

3. The tetrad field changes the (perceived) energy of waves such that they feel a force coupled to the stress-energy tensor. This is the Lorentz-covariant analog of the ordinary gravitational force. Quantitatively, it is the *torsion force* from teleparallel gravity. (Section 4.1–Section 4.2)

4. The combination of the above two effects implies the geodesic equation from general relativity. Also, one must rewrite physical laws using the metric tensor and covariant derivative, as in general relativity. (Section 4.3)

[1] Some authors use "teleparallel gravity" to refer to a family of theories, not all of which are equivalent to general relativity; we always use it to mean the teleparallel equivalent to general relativity (TEGR).

5. We next turn to the wave equation for the tetrad field, i.e., the Lorentz-covariant analog of (1). By demanding that this wave equation has a specific mathematical form—analogous to Maxwell's equations—we end up with a parameterized family of options. (Section 5.1–Section 5.2)

6. We observe that our theory so far is invariant with respect to a specific kind of global Lorentz transformation. Following existing work, we invoke a basic gauge principle: we require invariance under the corresponding *local* (spacetime-varying) Lorentz transformations. That narrows down our family of options to a single wave equation—the same wave equation that appears in teleparallel gravity and general relativity. (Section 5.3)

Remarks marked with an asterisk (***Remark**) describe connections to the traditional presentations of teleparallel gravity and general relativity. These remarks are intended for readers already familiar with those presentations. Otherwise, the paper requires knowledge only of special relativity (including Einstein summation notation), the covariant form of electromagnetism, and basic quantum mechanics.

1.2 Conventions

Our setting is ordinary (flat) Minkowski spacetime; gravity will be a classical field within it. This background spacetime is subject to the rules of special relativity. We set $c = 1$ and use the metric signature $(+, -, -, -)$.

Ordinary 4-vectors in the background spacetime are *spacetime vectors*. Section 3 will introduce a second type of 4-vector, *tangent-space vectors*. To distinguish them, we adopt the convention that spacetime vectors use Greek indices (v^μ, w_ν, etc.) while tangent-space vectors use Latin indices (v^a, w_b, etc.).

2 Perceived Time Dilation

We begin our motivation by postulating the existence of a gravitational potential $\phi(t, x) \leq 0$ satisfying the first property in the introduction: for any object with energy E, the field grants it a potential energy term $\phi(t, x)E$ (recalling $c = 1$).

For any quantum system, this potential energy shows up as a term

$$H_{\mathrm{gr}} := \phi(t, x)H$$

in the system's Hamiltonian, where H is the rest of the Hamiltonian for that system—used to measure its energy. The system's Schrödinger equation then becomes

$$\begin{aligned} \frac{\partial}{\partial t}\psi &= -iH_{\mathrm{total}}\psi \\ &= -i(H + H_{\mathrm{gr}})\psi \\ &= (1 + \phi(t, x))\,(-iH\psi). \end{aligned} \tag{2}$$

When $\phi(t, x)$ is static, this is mathematically equivalent to the "time-dilated" Schrödinger equation

$$\frac{\partial}{\partial t'}\psi = -iH\psi, \qquad t' := (1 + \phi(t, x))\,t.$$

Thus the gravitational potential $\phi(t, x)$ causes any system at (t, x) to evolve more slowly over time, by a factor of $1 + \phi(t, x) \leq 1$. In other words, it causes *perceived time dilation*: an observer at (t, x) will perceive time to pass at $1 + \phi(t, x)$ time its rate in the background spacetime.[2]

Note that we are not asserting any change to the nature of time itself in the background spacetime. The effect is purely mechanical, being implemented by a potential energy term in the Hamiltonian, like any other classical field. However, because gravitation is universal—interacting with all systems in proportion to their energy—its effect is observationally indistinguishable from true time dilation.

Actually, our perceived time dilation factor $1 + \phi(t, x)$ is only correct as a first approximation. It neglects the effect of time dilation on the gravitational field itself. To incorporate this self-interaction in a consistent way, divide $\phi(t, x)$ into arbitrarily small quanta $\phi(t, x)/n$ and apply each in series, yielding an exact perceived time dilation factor

$$\lim_{n\to\infty}\left(1 + \frac{\phi(t, x)}{n}\right)^n = e^{\phi(t,x)} \leq 1.$$

The exponential prevents this time dilation factor from becoming zero or negative.

***Remark 2.1.** Teleparallel gravity is traditionally presented as a gauge theory for the translation group, whose gauge generators are the 4-momentum operators $i\partial_a$, $a \in \{t, x, y, z\}$ Aldrovandi and Pereira (2013). In the Newtonian case above, the gauge covariant derivative for t is $D_t = \partial/\partial t - \phi\partial/\partial t$. Replacing $\partial/\partial t$ with D_t in the Schrödinger equation gives an equation equivalent to (2) (to first order in ϕ).

3 Tetrad Field

Consider an observer at (t, x) who is experiencing a perceived time dilation factor $e^{\phi(t,x)}$. When shown a displacement 4-vector written in spacetime coordinates as

[2] For a moving observer, special relativity's time dilation is layered on top of this effect. I.e., their proper time τ is a function of the gravitationally-dilated coordinate t' instead of t.

$(1 \text{ second}, 0, 0, 0)$, the observer will instead perceive the displacement 4-vector $(e^{\phi(t,x)} \text{ seconds}, 0, 0, 0)$.

More generally, for any spacetime displacement 4-vector $(\Delta t, \Delta x, \Delta y, \Delta z)$, the observer will instead perceive the 4-vector

$$\begin{pmatrix} e^{\phi(t,x)}\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} e^{\phi(t,x)} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}. \quad (3)$$

Call the matrix above the *tetrad* and denote it $h^a{}_\rho(t,x)$.

It is reasonable to assert that the same relation holds for arbitrary contravariant 4-vectors, not just the displacement vector. E.g., the perceived 4-velocity of a particle is also multiplied by $h^a{}_\rho(t,x)$. Thus $h^a{}_\rho(t,x)$ is a general transformation from a spacetime 4-vector v^ρ to the corresponding "perceived 4-vector"

$$v^a := h^a{}_\rho(t,x)v^\rho. \quad (4)$$

So far, we have only considered perceived time dilation. Now suppose we Lorentz transform both the background spacetime and the observer's perceived 4-vectors. In order for (4) to continue to hold, we see that the tetrad must transform covariantly, as a Lorentz matrix (i.e., a $(1,1)$-tensor).

Thus to make our theory Lorentz invariant, it is not sufficient to consider only the scalar potential $\phi(t,x)$ and perceived time dilation. Our theory must instead feature a more general Lorentz-matrix field $h^a{}_\rho(t,x)$ satisfying (4): the *tetrad field*.

Example 3.1. Suppose Alice and Bob are moving at speed v to the right, parallel to a nearby stationary wall of mass. At some point, Alice drifts closer to the wall of mass; this causes her gravitational potential to strengthen (become more negative), slowing her time evolution, and so her speed in spacetime coordinates decreases to $(1+\Delta\phi)v$.

Now consider Bob's perspective. Initially, Alice is stationary, while the wall is moving at speed v to the left. When Alice drifts closer to the wall, she begins moving in the leftward direction at speed $|(\Delta\phi)v|$. This is the *frame-dragging* effect from general relativity. It is the boosted form of perceived time dilation, and the tetrad field in Bob's frame is the boost of the matrix in (3).

Note that frame dragging is not a force; although Alice's spacetime velocity changes, her perceived velocity[3] remains constant, which is merely the law of inertia. However, unlike general relativity (but like teleparallel gravity), our theory will also include a true gravitational force, introduced in Section 4.

Our tetrad field is the same as the tetrad field in teleparallel gravity (Aldrovandi and Pereira, 2013), except that we have been using the term "perceived vector" for what teleparallel gravity calls *tangent-space vectors* (since they live within an alternative tangent space at each spacetime point). For the rest of the paper, we will adopt the teleparallel name. We also adopt the teleparallel convention that spacetime vector indices use Greek letters (v^μ, w_ν, etc.) while tangent-space vector indices use Latin letters (v^a, w_b, etc.), as in (4).

Remark 3.2. The name *tetrad* comes from treating $h^a{}_\rho$ as a set of four covector fields, one for each $a \in \{t,x,y,z\}$. In our interpretation, each covector field tells you the "actual" spacetime vectors corresponding to what an observer perceives to be the unit vector $e(a)$.

The name *teleparallelism* comes from the "distant parallelism" implied by the tetrad: one may define spacetime vectors at different points to be "parallel" if they have the same image under the tetrad. In other words, two spacetime vectors are parallel if observers at each point perceive them to be identical. This parallel structure (geometrically, a parallelizable manifold) arises naturally because we assumed an underlying flat spacetime.

3.1 Matrix Potential

Because the tetrad field is a Lorentz-matrix analog of the scalar time dilation factor $e^{\phi(t,x)}$, we might expect it to be related to a Lorentz-matrix analog $B^a{}_\rho(t,x)$ of the scalar potential $\phi(t,x)$, via the matrix exponential

$$h^a{}_\rho(t,x) = \exp\left(B^a{}_\rho(t,x)\right). \quad (5)$$

We will not actually use this "matrix potential" $B^a{}_\rho$, since it is only observable via the tetrad. However, the analogy suggests the following:

- The tetrad is always invertible, like all matrix exponentials.
- Generalizing (1), the tetrad ought to be coupled to the Lorentz-matrix analog of energy density: the stress-energy tensor $T^a{}_\rho$.

3.2 Usage

Let us briefly describe how to use the tetrad field, reproducing techniques from teleparallel gravity (de Andrade and Pereira, 1997).

We said in equation (4) that the tangent-space (perceived) form of a contravariant spacetime vector v^ρ is $v^a = h^a{}_\rho v^\rho$. In the reverse direction,

$$v^\rho = h_a{}^\rho v^a$$

where $h_a{}^\rho(t,x)$ is the *inverse tetrad*, defined as the matrix inverse of $h^a{}_\rho(t,x)$:

$$h^a{}_\rho h_b{}^\rho = \delta^a_b.$$

[3] According to observers positioned along her path who are stationary relative to the wall.

What about *covariant* 4-vectors w_ρ? It is reasonable to assume that the tetrad (i.e., generalized time dilation) does not change an observer's perception of Lorentz scalars, including the inner product $v^\rho w_\rho$ for any contravariant 4-vector v^ρ. Thus the tangent-space form w_a of w^ρ must satisfy $v^a w_a = v^\rho w_\rho$ for all v^ρ, which implies

$$w_a = w_\rho h_a{}^\rho, \qquad w_\rho = w_a h^a{}_\rho. \tag{6}$$

For a general tensor, one converts between spacetime and tangent-space indices by converting each index individually using the above rules.

We expect any physical law that holds in the absence of gravity to remain true only when written in terms of tangent-space indices. Indeed, an observer perceives the physical law to hold in tangent-space (perceived) form. To rewrite the physical law in terms of spacetime indices, we must convert each tensor index using the rules above. In particular:

- An inner product of contravariant vectors $\eta_{ab} u^a v^b$ becomes

$$\begin{aligned}\eta_{ab} u^a v^b &= \eta_{ab} \left(h^a{}_\mu u^\mu\right) \left(h^b{}_\nu u^\nu\right) \\ &= g_{\mu\nu} u^\mu v^\nu\end{aligned}$$

  where $g_{\mu\nu} := \eta_{ab} h^a{}_\mu h^b{}_\nu$ is the *metric tensor* from general relativity. It must be used in place of η_{ab} when lowering spacetime indices.

- We likewise have the inverse metric tensor $g^{\mu\nu} := \eta^{ab} h_a{}^\mu h_b{}^\nu$, used for the inner product of covectors and for raising spacetime indices.

- To convert the derivative $\partial_c v^a$, replace the input v^a with its equivalent $h^a{}_\sigma v^\sigma$ and multiply by $h_a{}^\rho$ to convert the output index a:

$$\begin{aligned}\partial_c v^a &\mapsto h_a{}^\rho h^c{}_\nu \partial_c \left(h^a{}_\mu v^\mu\right) \\ &= h_a{}^\rho \partial_\nu \left(h^a{}_\mu v^\mu\right) \\ &= h_a{}^\rho h^a{}_\mu \partial_\nu v^\mu + h_a{}^\rho \left(\partial_\nu h^a{}_\mu\right) v^\mu \\ &= \partial_\nu v^\rho + \tilde{\Gamma}^\rho_{\mu\nu} v^\mu\end{aligned}$$

  where

$$\tilde{\Gamma}^\rho_{\mu\nu} := h_a{}^\rho \left(\partial_\nu h^a{}_\mu\right) \tag{7}$$

  is the *Weitzenböck connection* from teleparallel gravity. One can derive similar formulas for covectors and general tensors, leading to the Weitzenböck covariant derivative; we omit its definition since we will modify this conversion rule anyway in Section 4.3.

***Remark 3.3.** In the translational gauge theory presentation of teleparallel gravity, the gauge covariant derivative is $D_a := h_a{}^\mu \partial_\mu \approx \partial_a - B_a{}^\mu \partial_\mu$, generalizing *Remark 2.1. Thus starting with a physical law that holds in the absence of gravity and is written in terms of ∂_a, the gauge prescription $\partial_a \to D_a$ is equivalent to the above conversion $\partial_a \to h_a{}^\mu \partial_\mu$.

More generally, one can think of any covector v_a as $v_a \approx \partial_a \psi$ for some function ψ. Thus $\partial_a \to D_a$ implies $v_a \to h_a{}^\mu \partial_\mu$. Here the significance of a vs μ is merely that v_a has not yet received the gauge prescription while v_μ has.

Note that the description here is dual to the one by Aldrovandi and Pereira (2013), who set $D_\mu = h^a{}_\mu \partial_a \approx \partial_\mu + B^a{}_\mu \partial_a$. Our description is needed to match the expected change in the Schrödinger equation from Section 2.

4 Torsion Force

Our goal in this section is to show that the tetrad field induces a true gravitational force, in addition to the inertial effects from Section 3.2. The resulting force is identical to the torsion force from teleparallel gravity, but motivated differently: we use mechanical principles—properties of waves—instead of gauge theory for the translation group.

Of course, we could start with the matrix potential $B^a{}_\rho(t, x)$ from Section 3.1 and treat it analogously to the electromagnetic potential. That leads to a force

$$\left(\partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu\right) T_a{}^\nu$$

on a particle with stress-energy tensor $T_a{}^\nu$, analogous to the Lorentz force law. The relation $h^a{}_\rho \approx \delta^a_\rho + B^a{}_\rho$ then suggests that we can express this force in terms of the tetrad as

$$\left(\partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu\right) T_a{}^\nu.$$

That answer is correct, but we will present an alternative, exact argument in the next two subsections.

Remark 4.1. For each $a_0 \in \{t, x, y, z\}$, $T_{a_0}{}^\nu$ is the flux density of one component of tangent-space (perceived) energy and momentum. It is analogous to current density but with electric charge replaced by a_0-momentum.

4.1 Newtonian Approximation

To start, we consider the special case of pure time dilation, corresponding to a scalar gravitational potential $\phi(t, x)$.

Suppose a wave travels from point A to point B, where the gravitational potential at A is 0 and the potential at B is $\phi(B) < 0$. Let the wave have frequency ν at point A. The number of peaks and troughs leaving A must equal the number arriving at point B, per unit of time *in the background spacetime*. Otherwise, peaks and troughs would "bunch up" in the region between A and B.

Thus at point B, the wave has frequency ν in spacetime coordinates, which corresponds to a *perceived* frequency

$$\frac{1}{e^{\phi(B)}}\nu \approx (1-\phi(B))\nu.$$

For a quantum particle, energy is proportional to frequency. Hence a quantum particle can increase its perceived energy from E to $\approx (1-\phi(B))E > E$ by traveling from A to B. Classically, this implies that the particle experiences a force in the B direction: the ordinary gravitational force.

Quantitatively, the force is $\approx -(\nabla\phi)E$, where E is the particle's energy. For a stationary massive particle, $E = m$, so the force is $\approx (-\nabla\phi)m$—the usual value from Newtonian gravity.

Remark 4.2. With explicit factors of c, the time dilation factor is $\approx 1 + \phi(t, x)/c^2$. Its $1/c^2$ is then canceled by the c^2 in $E = mc^2$, yielding a force $\approx (-\nabla\phi)m$. This explains how the gravitational force can be obvious in everyday life even though the corresponding time dilation factor is imperceptible.

4.2 Lorentz Covariance

Now we generalize the above argument to find the exact Lorentz-covariant force induced by a general tetrad field.

We argued in the previous section that a quantum particle's frequency should remain the same in spacetime coordinates when that particle moves to a point with different gravitational potential.

More generally, we argue that the coordinates of a free particle's stress-energy tensor should remain constant when written in the form

$$T_\rho{}^c,$$

i.e., with the first index in lowered spacetime form and the second in tangent-space form. Indeed, for a particle, we essentially have $T_\rho{}^c = p_\rho x^c$, where p_ρ is the 4-momentum and x^c is the 4-velocity.[4] 4-velocity remains the same in tangent-space (perceived) form because that is where the law of inertia applies. Meanwhile, $p_\rho \sim \partial_\rho\psi$ corresponds to the 4-frequency of the particle's de Broglie wave ψ, which remains constant in spacetime coordinates—again by counting peaks and troughs.

Now consider the trace of the particle's stress-energy tensor, $T = T^\rho{}_\rho$, which is a Lorentz scalar analogous to energy density. If the particle moves in direction μ, then T changes at the rate

$$\begin{aligned}\partial_\mu T &= \partial_\mu \left(T^\rho{}_\rho\right)\\ &= \partial_\mu \left(T_\rho{}^\rho\right)\\ &= \partial_\mu \left(h_c{}^\rho T_\rho{}^c\right)\\ &= \left(\partial_\mu h_c{}^\rho\right) T_\rho{}^c + h_c{}^\rho \left(\partial_\mu T_\rho{}^c\right)\\ &= \left(\partial_\mu h_c{}^\rho\right) T_\rho{}^c\end{aligned}$$

since $\partial_\mu T_\rho{}^c = 0$ by the previous paragraph. This ability to change T by moving implies the existence of a force

$$\begin{aligned}f'_\mu &= -\partial_\mu T\\ &= -\left(\partial_\mu h_c{}^\rho\right) T_\rho{}^c\\ &= h_a{}^\rho \left(\partial_\mu h^a{}_\nu\right) h_c{}^\nu T_\rho{}^c\\ &= \left(\partial_\mu h^a{}_\nu\right) T_a{}^\nu.\end{aligned}$$

The above relation is a Lorentz-covariant generalization of the Newtonian gravitational force, but it is not yet correct. The force $\left(\partial_\mu h^a{}_\nu\right) T_a{}^\nu$ is analogous to the first term $\left(\partial_\mu A_\nu\right) J^\nu$ appearing in the Lorentz force law from electromagnetism, where A_ν is the vector potential and J^ν is the 4-current density. We know that the complete Lorentz force law is

$$f_\mu^{\mathrm{EM}} = \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) J^\nu.$$

Here the antisymmetrization over μ and ν is necessary to preserve the norm of 4-velocity: we need

$$\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) x^\mu x^\nu = 0$$

for a particle with 4-velocity x^μ, since

$$\begin{aligned}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) x^\mu x^\nu &\propto x^\mu f_\mu^{\mathrm{EM}}\\ &= x^\mu \left(\frac{d}{dt} p_\mu\right)\\ &\propto x^\mu \left(\frac{d}{dt} x_\mu\right)\\ &= \frac{1}{2}\frac{d}{dt}\left(x^\mu x_\mu\right) = 0.\end{aligned}$$

Analogously, we argue that the correct force law for the tetrad field is

$$f_\mu = T^a{}_{\mu\nu} T_a{}^\nu, \qquad T^a{}_{\mu\nu} := \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu \tag{8}$$

where we have antisymmetrized over μ and the 4-velocity coordinate ν.

Converting the a coordinate to a spacetime coordinate in the usual way, we have

$$\begin{aligned}T^\rho{}_{\mu\nu} &= h_a{}^\rho T^a{}_{\mu\nu}\\ &= h_a{}^\rho \partial_\mu h^a{}_\nu - h_a{}^\rho \partial_\nu h^a{}_\mu\\ &= \tilde{\Gamma}^\rho_{\nu\mu} - \tilde{\Gamma}^\rho_{\mu\nu}.\end{aligned}$$

This is the definition of the *torsion tensor for the Weitzenböck connection*.[5] Thus our force is identical to the torsion force from teleparallel gravity Aldrovandi and Pereira (2013).

[4] Technically, we ought to spread the particle's mass over a small region to get a stress-energy *density*, then integrate over that region later to get the total force on the particle.

[5] Some authors define torsion to be the negative of this.

***Remark 4.3.** The above force is the usual Yang-Mills force derived from $[D_\mu, D_\nu]$, since the translation group is abelian.

4.3 Geodesic Equation and Covariant Derivative

The tetrad field and the torsion force together imply the usual laws of motion from general relativity: the geodesic equation, and the need to convert ordinary derivatives to covariant derivatives in physical laws. The argument comes from teleparallel gravity (de Andrade and Pereira, 1997); we summarize it here to keep the paper self-contained.

Consider a free particle with tangent-space momentum p^b and spacetime 4-velocity x^ν. As it moves in its direction of travel x^ν, its momentum changes due to the gravitational force:[6]

$$x^\nu \partial_\nu p^b = f^b = T^{ab}{}_\nu T_a{}^\nu = T^{ab}{}_\nu p_a x^\nu.$$

Rewriting this in spacetime coordinates gives

$$x^\nu \partial_\nu p^\rho + \tilde{\Gamma}^\rho_{\mu\nu} p^\mu x^\nu = T^{\mu\rho}{}_\nu p_\mu x^\nu$$

recalling our conversion rule for derivatives from Section 3.2.

We can equivalently write

$$x^\nu \partial_\nu p^\rho + \Gamma^\rho_{\mu\nu} p^\mu x^\nu = 0 \tag{9}$$

where

$$\Gamma^\rho_{\mu\nu} := \tilde{\Gamma}^\rho_{(\mu\nu)} - T_{(\mu}{}^\rho{}_{\nu)}\,.$$

Here the symmetrization over $\mu\nu$ makes no difference because p^μ and x^ν are parallel.

Expanding in terms of the tetrad shows that $\Gamma^\rho_{\mu\nu}$ is identical to the *Levi-Cevita connection*

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu} \right)$$

used in general relativity. In particular, our free-particle equation of motion (9) is equivalent to the geodesic equation from general relativity.

More generally, one may assert that for an object moving freely in the direction x^ν with an associated vector v^ρ (e.g., the axis of a spinning top), gravitation causes v^ρ to change at a rate

$$x^\nu \partial_\nu v^\rho + \Gamma^\rho_{\mu\nu} v^\mu x^\nu = 0, \tag{10}$$

analogous to the change in momentum. This makes sense when the associated vector is itself defined in terms of particles obeying (9). However, it is admittably questionable for the intrinsic spin of a quantum particle.

[6] Again, we technically ought to spread the particle's mass into a mass density and integrate at the end.

***Remark 4.4.** Some variants of teleparallel gravity do include a coupling to fermionic spin, e.g., Hayashi and Shirafuji (1979). However, it is more common in Poincaré gauge theories such as Kibble (1961) or Hayashi and Shirafuji (1980).

Even more generally, when an object is subject to non-gravitational forces, they will appear on the right-hand side of (10). We can then account for gravitation by taking the other forces' physical laws and replacing each partial derivative $\partial_\nu v^\rho$ with the *covariant derivative*

$$\nabla_\nu v^\rho := \partial_\nu v^\rho + \Gamma^\rho_{\mu\nu} v^\mu.$$

This modifies our conversion rule from Section 3.2 to account for the torsion force, in addition to the inertial effects induced by the tetrad.

One can similarly argue that for a covector w_μ, we should convert

$$\partial_\nu w_\mu \mapsto \nabla_\nu w_\mu := \partial_\nu w_\mu - \Gamma^\rho_{\mu\nu} w_\rho.$$

For a general spacetime tensor, one adds a separate $\Gamma^\rho_{\mu\nu}$ term for each tensor index.

In summary, given a physical law that holds in the absence of gravity, to make it also hold in the presence of gravity, we must perform the substitutions:

$$\eta_{\mu\nu} \to g_{\mu\nu}, \qquad \eta^{\mu\nu} \to g^{\mu\nu}, \qquad \partial_\mu \to \nabla_\mu.$$

This is the same prescription used in general relativity.

Remark 4.5. One must be careful because unlike ordinary derivatives, covariant derivatives do not commute: $\nabla_\mu \nabla_\nu \neq \nabla_\nu \nabla_\mu$. In particular, there is an ambiguity when applying the above substitutions to $\partial_\mu \partial_\nu$; this causes non-obvious changes to e.g. Maxwell's equations in the Lorenz gauge. However, covariant derivatives do commute with the metric tensor (*metric compatibility*), so there is no ambiguity when raising and lowering spacetime coordinates in the presence of covariant derivatives.

See Misner et al. (1973) for an introduction to covariant derivatives in the context of general relativity.

4.4 Covariant Derivatives and Tangent-Space Indices

It is convenient to introduce a notation where ∇_μ can be applied to tensors that have tangent-space indices, not just spacetime indices. In this notation, ∇_μ ignores the tangent-space indices, i.e., it does not add a $\Gamma^\rho_{\mu\nu}$ term for them. For example,

$$\nabla_\sigma T_a{}^{\nu\sigma} := \partial_\sigma T_a{}^{\nu\sigma} + \Gamma^\nu_{\lambda\sigma} T_a{}^{\lambda\sigma} + \Gamma^\sigma_{\lambda\sigma} T_a{}^{\nu\lambda}.$$

Remark 4.6. We can rewrite our force tensor (8) as

$$T^a{}_{\mu\nu} = \nabla_\mu h^a{}_\nu - \nabla_\nu h^a{}_\mu \tag{11}$$

because $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$. Thus the substitution $\partial_\mu \to \nabla_\mu$ does not change (8).

This is only a notational convention, not a statement about covariance under general coordinate transformations (a property possessed by ∇_μ that explains its name). In (11), the tangent-space indices are only covariant under global Lorentz transformations. However, it is possible to convert equations like (11) into truly covariant equations involving no tangent-space indices, using the contorsion tensor—see Appendix A.1.

5 Wave Equation

The final component of teleparallel gravity is its wave equation. It relates the tetrad field to its source, the stress-energy tensor, providing the relativistic analog of (1).

5.1 First Attempt

By analogy with electromagnetism, we might naïvely expect the wave equation to be

$$\nabla_\sigma T_a{}^{\nu\sigma} = k\overline{T}_a{}^{\nu}. \tag{12}$$

Here $k = 8\pi G$ is a constant setting the field strength, while $\overline{T}_a{}^{\nu}$ is the stress-energy tensor for all systems *including* the gravitational field itself.

By the same proof as for Maxwell's equations (Misner et al., 1973, §22.4) and our convention that the covariant derivative ignores tangent-space indices, we have the algebraic relation

$$\nabla_\nu \nabla_\sigma T_a{}^{\nu\sigma} = 0. \tag{13}$$

Thus (12) implies $\nabla_\nu \overline{T}_a{}^{\nu} = 0$, the conservation law for tangent-space energy and momentum.

Rearrange 12 as

$$h_b{}^{\mu}\eta^{ab}\nabla_\sigma T_a{}^{\nu\sigma} - kt_T^{\mu\nu} = kT^{\mu\nu}, \tag{14}$$

where $t_T^{\mu\nu}$ is the stress-energy tensor of the gravitational field itself and $T^{\mu\nu}$ is the stress-energy tensor of matter.[7] Again by analogy with electromagnetism, we might expect

$$t_T^{\mu\nu} = \frac{1}{k}\left(T^{a\lambda\mu}T_{a\lambda}{}^{\nu} - \frac{1}{4}g^{\mu\nu}T^{a\lambda\tau}T_{a\lambda\tau}\right). \tag{15}$$

Then 14 becomes

$$h_b{}^{\mu}\eta^{ab}\nabla_\sigma T_a{}^{\nu\sigma} - T^{a\lambda\mu}T_{a\lambda}{}^{\nu} + \frac{1}{4}g^{\mu\nu}T^{a\lambda\tau}T_{a\lambda\tau} = kT^{\mu\nu}. \tag{16}$$

This naïve wave equation is a second-order partial differential equation relating $h^a{}_\rho$ to $T^{\mu\nu}$, just like Einstein's equation from general relativity.[8] Unfortunately, it is not correct: it is not equivalent to Einstein's equation, and it does not admit the well-tested Schwarzschild solution.

[7] I.e., non-gravitational matter and energy.

[8] After expanding the metric tensor in terms of the tetrad: $g_{\mu\nu} = \eta_{ab}h^a{}_\mu h^b{}_\nu$.

***Remark 5.1.** Equation (12) is the Yang-Mills wave equation for the translation gauge theory, since the translation group is abelian.

5.2 Parameterized Equation

Let us search for a more general "Maxwell-like" wave equation. We still want it to have the form

$$\nabla_\sigma E_a{}^{\nu\sigma} = k\overline{T}_a{}^{\nu} \tag{17}$$

for some tensor $E_a{}^{\nu\sigma}$ that is antisymmetric in ν and σ; this ensures $\nabla_\nu \overline{T}_a{}^{\nu} = 0$, analogous to (13). To keep it grounded in the existing theory, $E_a{}^{\nu\sigma}$ should be a linear function of the torsion tensor (our theory's force tensor). However, we allow it to rearrange coordinates and use additional factors of the tetrad.

The tensor $E_a{}^{\nu\sigma}$ must then have the form

$$\begin{aligned} E_a{}^{\nu\sigma} &= 2aT_a{}^{\nu\sigma} + b\left(T^{\sigma\nu}{}_a - T^{\nu\sigma}{}_a\right) \\ &\quad + c\left(h_a{}^{\sigma}T^{\lambda\nu}{}_\lambda - h_a{}^{\nu}T^{\lambda\sigma}{}_\lambda\right). \end{aligned} \tag{18}$$

for some dimensionless constants a, b, c (Aldrovandi and Pereira, 2013, Ch. 8). Note that the constants only matter up to an overall scale, since they can trade off with the field strength k.

The wave equation in terms of $T^{\mu\nu}$ is then

$$h_b{}^{\mu}\eta^{ab}\nabla_\sigma E_a{}^{\nu\sigma} - kt_E^{\mu\nu} = kT^{\mu\nu}, \tag{19}$$

where $t_E^{\mu\nu}$—the stress-energy tensor of the gravitational field itself—is yet to be determined.

Remark 5.2. New General Relativity (Hayashi and Shirafuji, 1979) is a variant of teleparallel gravity that has a parametric Lagrangian, somewhat like the parameterization here. Its creators propose to determine the parameters by experiment. Aldrovandi and Pereira (2013, Ch. 9.6) note that only one choice of their parameters—the one yielding Einstein's equation—admits the Schwarzschild solution, and most other choices are ruled out by solar system experiments.

The $f(T)$ theories of gravity consider Lagrangians drawn from a different family of functions of the torsion tensor. They aim to modify general relativity for cosmological reasons (Cai et al., 2016).

5.3 Local Lorentz Invariance

It remains to determine the constants a, b, c. Of course, we could by fiat choose the values that lead to Einstein's equation from general relativity (the "correct" answer); however, this is unsatisfying.

Instead, observe that our theory so far is invariant with respect to global Lorentz transformations of the tangent-space vectors. This means that if we multiply each tangent-space vector v^b by a fixed Lorentz matrix

$\Lambda^a{}_b$, then all of our equations still hold. For example, the force tensor $T^a{}_{\mu\nu}$ transforms covariantly:

$$\begin{aligned} T^a{}_{\mu\nu} \to T^a{}_{\mu\nu}{}' &= \partial_\mu \left(h^a{}_\nu{}'\right) - \partial_\nu \left(h^a{}_\mu{}'\right) \\ &= \partial_\mu \left(\Lambda^a{}_b h^b{}_\nu\right) - \partial_\nu \left(\Lambda^a{}_b h^b{}_\mu\right) \\ &= \Lambda^a{}_b \left(\partial_\mu h^b{}_\nu - \partial_\nu h^b{}_\mu\right) \\ &= \Lambda^a{}_b T^b{}_{\mu\nu}. \end{aligned}$$

Hence the force law (8) is invariant. This works because we constructed the theory to be compatible with special relativity, though note that invariance holds despite that we are not simultaneously Lorentz transforming the spacetime coordinates.

Gauge theory is a general technique for constructing field theories by "promoting" a global invariance to a local (spacetime-varying) invariance. In our case, this means that instead of requiring invariance with respect to a global Lorentz transformation Λ, we also require invariance with respect to a *local* Lorentz transformation $\Lambda(t, x)$ that is different at each point in spacetime.

The local nature of $\Lambda(t, x)$ implies that it no longer commutes with spacetime derivatives. For example, $T^a{}_{\mu\nu}$ no longer transforms covariantly, so our force law (8) is not local Lorentz invariant. (However, the metric tensor is invariant, hence also the geodesic equation (9) and covariant derivative—see the proof of Theorem 5.5 below.)

Nonetheless, let us now enforce a basic version of the gauge principle, by requiring the wave equation in particular to be invariant with respect to local Lorentz transforms of the tangent-space vectors (*local Lorentz invariance*).

***Remark 5.3.** In some formulations of teleparallel gravity, a second field is introduced, called the spin connection or purely inertial connection Aldrovandi and Pereira (2013). It transforms together with the tetrad to make a theory that is local Lorentz invariant for any choice of $E_a{}^{\nu\sigma}$.

See Golovnev et al. (2017) for a comparison of that "covariant" formulation to the "pure-tetrad" formulation used here, in which local Lorentz invariance is a nontrivial constraint (a well-known fact). Those authors object to the pure-tetrad formulation because it chooses a specific gauge—the one in which the spin connection is zero. We are comfortable with that choice because it arises naturally from our flat background spacetime.

Let $S_a{}^{\nu\sigma}$ be the value of $E_a{}^{\nu\sigma}$ resulting from parameters $a = \frac{1}{4}$, $b = \frac{1}{2}$, $c = -1$:

$$\begin{aligned} S_a{}^{\nu\sigma} = &\frac{1}{2} T_a{}^{\nu\sigma} + \frac{1}{2}\left(T^{\sigma\nu}{}_a - T^{\nu\sigma}{}_a\right) \\ &- \left(h_a{}^\sigma T^{\lambda\nu}{}_\lambda - h_a{}^\nu T^{\lambda\sigma}{}_\lambda\right). \end{aligned} \tag{20}$$

This is called the *superpotential* for teleparallel gravity Aldrovandi and Pereira (2013).

Define the stress-energy tensor of the gravitational field itself to be

$$t_S^{\mu\nu} = \frac{1}{k}\left(T^{a\lambda\mu} S_{a\lambda}{}^\nu - \frac{1}{4} g^{\mu\nu} T^{a\lambda\tau} S_{a\lambda\tau}\right). \tag{21}$$

This is still similar to the electromagnetic stress-energy tensor, except unlike in (15), one copy of the force tensor is replaced with the superpotential. We will validate this expression in Section 5.4.

Our wave equation (19) is then

$$h_b{}^\mu \eta^{ab} \nabla_\sigma S_a{}^{\nu\sigma} - T^{a\lambda\mu} S_{a\lambda}{}^\nu + \frac{1}{4} g^{\mu\nu} T^{a\lambda\tau} S_{a\lambda\tau} = k T^{\mu\nu}. \tag{22}$$

***Remark 5.4.** In the translational gauge theory presentation, the above wave equation and $t_S^{\mu\nu}$ (which is the spacetime form of the gauge current $J_a{}^\nu$) are derived from the teleparallel Lagrangian density[9]

$$\frac{1}{4k} h T^{a\lambda\tau} S_{a\lambda\tau},$$

where $h = \det(h^a{}_\rho) = \sqrt{-g}$. This is like the Yang-Mills Lagrangian except that one copy of the force tensor is replaced with the superpotential. It is not local Lorentz invariant, but it is equivalent to (differs by a divergence from) the Einstein-Hilbert Lagrangian R, which is invariant—hence so is the wave equation Aldrovandi and Pereira (2013).

Theorem 5.5. *The left-hand side of (22) can be written in terms of the metric tensor and its inverse, using only ordinary derivatives and algebraic operations. Hence it is local Lorentz invariant.*

Specifically, it equals $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$, the Einstein tensor from general relativity. It follows that the wave equation (19) is identical to Einstein's equation.

Proof. For the equivalence with Einstein's tensor (and that tensor's usual definition in terms of the metric tensor and its inverse), see Appendix A.

Gauge invariance follows because the metric tensor and its inverse are local Lorentz invariant: for any local Lorentz transformation $\Lambda(t, x)$,

$$\begin{aligned} g_{\mu\nu}{}' &= h^a{}_\mu{}' \eta_{ab} h^b{}_\nu{}' \\ &= h^c{}_\mu \Lambda(t,x)^a{}_c \eta_{ab} \Lambda(t,x)^b{}_d h^d{}_\nu \\ &= h^c{}_\mu \eta_{cd} h^d{}_\nu \\ &= g_{\mu\nu} \end{aligned}$$

and likewise for $g^{\mu\nu}$. □

Theorem 5.6. *The superpotential $S_a{}^{\nu\sigma}$ is the unique choice of $E_a{}^{\nu\sigma}$ (up to scale) for which there exists a $t_E^{\mu\nu}$ such that:*

[9] Or "TEGR Lagrangian density", in works that consider other possible Lagrangians.

1. $t_E^{\mu\nu}$ *is an algebraic function of the torsion tensor and the tetrad.*
2. *The left-hand side of the wave equation (19),*

$$h_b{}^{\mu}\eta^{ab}\nabla_\sigma E_a{}^{\nu\sigma} - kt_E^{\mu\nu},$$

is local Lorentz invariant.

Proof sketch. We already proved that $S_a{}^{\nu\sigma}$ has this property.

To show uniqueness, consider the transformation of the specific tetrad $h^a{}_\rho = \delta^a_\rho$ by a local Lorentz transform $\Lambda(t,x)$ satisfying $\Lambda(0,0)^a{}_b = \delta^a_b$ and $\partial_\sigma\Lambda(0,0) = 0$. With μ lowered, the left-hand side of (19) changes by[10]

$$\begin{aligned}&2a\partial^\sigma\partial_\sigma\Lambda_\mu{}^\nu - 2a\partial_\sigma\partial^\nu\Lambda_\mu{}^\sigma\\ &+ b\partial^\sigma\partial_\sigma\Lambda^\nu{}_\mu - b\partial^\sigma\partial_\mu\Lambda^\nu{}_\sigma - b\partial_\sigma\partial^\nu\Lambda^\sigma{}_\mu + b\partial^\sigma\partial_\mu\Lambda_\sigma{}^\nu\\ &+ c\delta^\mu_\nu\partial^\sigma\partial_\sigma\Lambda^\lambda{}_\lambda - c\delta^\mu_\nu\partial^\sigma\partial_\sigma\Lambda^\lambda{}_\lambda - c\partial_\mu\partial^\nu\Lambda^\sigma{}_\sigma + c\partial_\mu\partial^\sigma\Lambda_\sigma{}^\nu,\end{aligned}$$

which thus needs to be 0.

We can combine and eliminate terms using the relation

$$\begin{aligned}0 &= \partial_\alpha\partial_\beta\delta^\gamma_\delta\\ &= \partial_\alpha\partial_\beta\left(\Lambda^\gamma{}_\varepsilon\Lambda_\varphi{}^\varepsilon\right)\\ &= \left(\partial_\alpha\partial_\beta\Lambda^\gamma{}_\varepsilon\right)\Lambda_\varphi{}^\varepsilon + \Lambda^\delta{}_\varepsilon\left(\partial_\alpha\partial_\beta\Lambda_\varphi{}^\varepsilon\right)\\ &= \partial_\alpha\partial_\beta\Lambda^\delta{}_\varphi + \partial_\alpha\partial_\beta\Lambda_\varphi{}^\delta.\end{aligned}$$

The resulting constraint is

$$\begin{aligned}0 &= (b-2a)\left(\partial^\sigma\partial_\sigma\Lambda^\nu{}_\mu - \partial^\nu\partial_\sigma\Lambda^\sigma{}_\mu\right)\\ &\quad + (-2b-c)\left(\partial^\sigma\partial_\mu\Lambda^\nu{}_\sigma\right).\end{aligned} \tag{23}$$

Finally, there exist specific values of $\Lambda(t,x)$, satisfying our conditions above, for which (23) yields $b-2a=0$ and $-2b-c=0$:

- The boost in $+\hat{x}$ with rapidity $\sinh^{-1}(t^2)$, evaluated at $\mu = x, \nu = t$.
- The boost in $+\hat{x}$ with rapidity $\cosh^{-1}(t^2+1)$, evaluated at $\mu = x, \nu = t$.

Solve to find that (a,b,c) must be a scalar multiple of the superpotential's $\left(\frac{1}{4}, \frac{1}{2}, -1\right)$. □

***Remark 5.7.** Lucas and Pereira (2008) consider the same choices for $E_a{}^{\nu\sigma}$ as here (though our notation is closer to Aldrovandi and Pereira (2013, Ch. 8)). They justify the choice of superpotential by showing that the map

$$T^a{}_{\mu\nu} \to \frac{h}{2}\varepsilon_{\mu\nu\alpha\beta}S^{a\alpha\beta}$$

has similar properties to the Hodge dual $\star$ used in gauge theory—specifically, it is anti-self-dual. However, the superpotential is not unique in this regard (Horňák, 2024).

[10] We ignore the distinction between tangent-space and spacetime indices, since $h^a{}_\rho = \delta^a_\rho$.

***Remark 5.8.** Our appeal to Lorentz gauge symmetry suggests that what we actually want is not a gauge theory for the translation group, but instead the Poincaré group, which is generated by translations and Lorentz transformations. Poincaré gauge theories have been pursued before (see Hehl (2023) for a recent overview) and are closely related to teleparallel gravity, but we are not aware of any that uniquely derive the teleparallel wave equation. Instead, they either choose the Einstein-Hilbert Lagrangian by fiat or consider a parameterized family of Lagrangians generalizing $E_a{}^{\nu\sigma}$.

5.4 Gravitational Stress-Energy Tensor

It is well-known that

$$\nabla_\nu\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0.$$

Then by Theorem 5.5, our wave equation (22) implies $\nabla_\nu T^{\mu\nu} = 0$. That is, non-gravitational energy and momentum are conserved when written in spacetime coordinates, as in general relativity. Likewise, the wave equation implies $T^{\mu\nu} = T^{\nu\mu}$.

Recall that our wave equation is equivalent to

$$\nabla_\sigma S_a{}^{\nu\sigma} = k\overline{T}_a{}^\nu,$$

which is just (17) with $E_a{}^{\nu\sigma} \to S_a{}^{\nu\sigma}$. This form of the equation implies $\nabla_\nu\overline{T}_a{}^\nu = 0$ (Section 5.2).

From these facts, one derives the following result, which is analogous to the conservation law for the electromagnetic stress-energy tensor (Misner et al., 1973, §22.4). It shows that our choice of the gravitational stress-energy tensor $t_S^{\mu\nu}$ in (21) was correct.

Theorem 5.9. *Let* $f^b = T^{\mu b\nu}T_{\mu\nu}$ *be the total gravitational force on matter at a point. Then*

$$f^b + \nabla_\nu t_S^{b\nu} = 0. \tag{24}$$

That is, the tangent-space energy and momentum added to matter by the gravitational force balances the change in gravity's own tangent-space energy and momentum density.

The proof is in Appendix A.

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A Additional Proofs for Section 5

This section contains proofs that were omitted from Section 5. They are straightforward but tedious algebraic arguments.

In principle, each equation below can be proved by expanding both sides in terms of the tetrad. We instead give shorter proofs that expand in terms of the contorsion tensor. They are based on the exposition in Aldrovandi and Pereira (2013).

A.1 Contorsion Tensor

The *contorsion tensor* for the Weitzenböck connection is defined by

$$K^{\rho}{}_{\mu\nu} := \tilde{\Gamma}^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}, \tag{25}$$

where $\tilde{\Gamma}^{\rho}_{\mu\nu}$ is the Weitzenböck connection from (7) and $\Gamma^{\rho}_{\mu\nu}$ is the Levi-Cevita connection. The contorsion tensor is interesting for our purposes because

$$\begin{aligned} h_a{}^{\rho}\nabla_{\nu}h^a{}_{\mu} &= h_a{}^{\rho}\partial_{\nu}h^a{}_{\mu} - h_a{}^{\rho}\Gamma^{\lambda}_{\mu\nu}h^a{}_{\lambda} \\ &= \tilde{\Gamma}^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu} \\ &= K^{\rho}{}_{\mu\nu} \end{aligned} \tag{26}$$

using the notation from Section 4.4.

Let us mention three useful identities for $K_{\rho\mu\nu}$. First, our rewritten force law (11) implies

$$T_{\rho\mu\nu} = K_{\rho\nu\mu} - K_{\rho\mu\nu}. \tag{27}$$

Second, it is antisymmetric in the first two indices (expand both sides in terms of the tetrad):

$$K_{\rho\mu\nu} = -K_{\mu\rho\nu}. \tag{28}$$

Third, it can be expressed in terms of the torsion tensor (use (27) and (28)):

$$K_{\rho\mu\nu} = \frac{1}{2}\left(T_{\mu\rho\nu} + T_{\nu\rho\mu} - T_{\rho\mu\nu}\right). \tag{29}$$

Using these relations, we can rewrite the superpotential (20) as

$$\begin{aligned} S_a{}^{\nu\sigma} &= K^{\nu\sigma}{}_a - \left(h_a{}^{\sigma}T^{\lambda\nu}{}_{\lambda} - h_a{}^{\nu}T^{\lambda\sigma}{}_{\lambda}\right) \\ &= K^{\nu\sigma}{}_a + h_a{}^{\sigma}K^{\lambda\nu}{}_{\lambda} - h_a{}^{\nu}K^{\lambda\sigma}{}_{\lambda}. \end{aligned} \tag{30}$$

A.2 Equivalence with Einstein's Tensor (Theorem 5.5)

The *Riemann tensor* is defined in terms of the Levi-Cevita connection (hence also the metric tensor and its inverse) by

$$R^{\rho}{}_{\theta\mu\nu} := \partial_{\mu}\Gamma^{\rho}_{\theta\nu} - \partial_{\nu}\Gamma^{\rho}_{\theta\mu} + \Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\theta\nu} - \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\theta\mu}. \tag{31}$$

The *Ricci curvature* is $R_{\mu\nu} := R^{\sigma}{}_{\mu\sigma\nu}$, and the *Ricci scalar* is $R := R^{\tau}{}_{\tau}$. It is well-known that the Ricci curvature is symmetric: $R_{\mu\nu} = R_{\nu\mu}$.

We can also consider the Riemann tensor for the Weitzenböck connection,

$$\tilde{R}^{\rho}{}_{\theta\mu\nu} := \partial_\mu \tilde{\Gamma}^{\rho}_{\theta\nu} - \partial_\nu \tilde{\Gamma}^{\rho}_{\theta\mu} + \tilde{\Gamma}^{\rho}_{\sigma\mu}\tilde{\Gamma}^{\sigma}_{\theta\nu} - \tilde{\Gamma}^{\rho}_{\sigma\nu}\tilde{\Gamma}^{\sigma}_{\theta\mu}.$$

Expanding in terms of the tetrad shows that $\tilde{R}^{\rho}{}_{\theta\mu\nu} = 0$.

***Remark A.1.** Geometrically, the vanishing of $\tilde{R}^{\rho}{}_{\theta\mu\nu}$ indicates that the Weitzenböck connection has no curvature. Indeed, the corresponding manifold is parallelizable due to the tetrad (Remark 3.2).

By substituting $\Gamma^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} - K^{\rho}{}_{\mu\nu}$ into 31 and using $\tilde{R}^{\rho}{}_{\theta\mu\nu} = 0$ and $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$, one finds (De Andrade et al., 2002)

$$-R^{\rho}{}_{\theta\mu\nu} = \nabla_\mu K^{\rho}{}_{\theta\nu} - \nabla_\nu K^{\rho}{}_{\theta\mu} + K^{\sigma}{}_{\theta\nu}K^{\rho}{}_{\sigma\mu} - K^{\sigma}{}_{\theta\mu}K^{\rho}{}_{\sigma\nu}. \quad (32)$$

Proof of Theorem 5.5. It remains to show that

$$\begin{aligned} &h_b{}^{\mu}\eta^{ab}\nabla_\sigma S_a{}^{\nu\sigma} - T^{a\lambda\mu}S_{a\lambda}{}^{\nu} + \frac{1}{4}g^{\mu\nu}T^{a\lambda\tau}S_{a\lambda\tau} \\ &\quad = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R. \end{aligned}$$

Since the Ricci curvature is symmetric, we can change the right-hand side to $R^{\nu\mu} - \frac{1}{2}g^{\mu\nu}R$. Substitute (32) to write it in terms of $K^{\rho}{}_{\mu\nu}$ and covariant derivatives:

$$\begin{aligned} &\text{RHS} = R^{\nu\mu} - \frac{1}{2}g^{\mu\nu}R \\ &= -\left(\nabla_\sigma K^{\sigma\nu\mu} - \nabla^{\mu}K^{\sigma\nu}{}_{\sigma} + K^{\lambda\nu\mu}K^{\sigma}{}_{\lambda\sigma} - K^{\lambda\nu\sigma}K_{\sigma\lambda}{}^{\mu}\right) \\ &+ \frac{1}{2}g^{\mu\nu}\big(\nabla_\sigma K^{\sigma\tau}{}_{\tau} - \nabla_\tau K^{\sigma\tau}{}_{\sigma} + K^{\lambda\tau}{}_{\tau}K^{\sigma}{}_{\lambda\sigma} \\ &\quad - K^{\lambda\tau}{}_{\sigma}K^{\sigma}{}_{\lambda\tau}\big) \\ &= -\nabla_\sigma K^{\sigma\nu\mu} + \nabla^{\mu}K^{\sigma\nu}{}_{\sigma} - K^{\lambda\nu\mu}K^{\sigma}{}_{\lambda\sigma} - K^{\nu\lambda\sigma}K_{\sigma\lambda}{}^{\mu} \\ &- g^{\mu\nu}\nabla_\tau K^{\sigma\tau}{}_{\sigma} - \frac{1}{2}g^{\mu\nu}K^{\tau\lambda}{}_{\tau}K^{\sigma}{}_{\lambda\sigma} + \frac{1}{2}g^{\mu\nu}K^{\tau\lambda\sigma}K_{\sigma\lambda\tau}. \end{aligned}$$

For the left-hand side, start by expanding

$$\begin{aligned} h_b{}^{\mu}\eta^{ab}\nabla_\sigma S_a{}^{\nu\sigma} &= h_b{}^{\mu}\eta^{ab}\nabla_\sigma\left(h_a{}^{\lambda}S_\lambda{}^{\nu\sigma}\right) \\ &= h_b{}^{\mu}\eta^{ab}\left(K_a{}^{\lambda}{}_{\sigma}S_\lambda{}^{\nu\sigma} + h_a{}^{\lambda}\nabla_\sigma S_\lambda{}^{\nu\sigma}\right) \\ &= K^{\mu\lambda}{}_{\sigma}S_\lambda{}^{\nu\sigma} + \nabla_\sigma S^{\mu\nu\sigma}. \end{aligned}$$

Use this relation together with (27) and (30) to write the left-hand side in terms of $K^{\rho}{}_{\mu\nu}$ and covariant derivatives:

$$\begin{aligned} &\text{LHS} \\ &= K^{\mu\lambda}{}_{\sigma}S_\lambda{}^{\nu\sigma} + \nabla_\sigma S^{\mu\nu\sigma} - T^{a\lambda\mu}S_{a\lambda}{}^{\nu} + \frac{1}{4}g^{\mu\nu}T^{a\lambda\tau}S_{a\lambda\tau} \\ &= K^{\mu\lambda}{}_{\sigma}\left(K^{\nu\sigma}{}_{\lambda} + \delta^{\sigma}_{\lambda}K^{\tau\nu}{}_{\tau} - \delta^{\nu}_{\lambda}K^{\tau\sigma}{}_{\tau}\right) \\ &+ \nabla_\sigma\left(K^{\nu\sigma\mu} + g^{\mu\sigma}K^{\lambda\nu}{}_{\lambda} - g^{\mu\nu}K^{\lambda\sigma}{}_{\lambda}\right) \\ &- \left(K^{a\mu\lambda} - K^{a\lambda\mu}\right)\left(K_\lambda{}^{\nu}{}_{a} + h_a{}^{\nu}K^{\tau}{}_{\lambda\tau} - h_a{}^{\omega}g_{\omega\lambda}K^{\tau\nu}{}_{\tau}\right) \\ &+ \frac{1}{4}g^{\mu\nu}\left(K^{a\tau\lambda} - K^{a\lambda\tau}\right)\big(K_{\lambda\tau a} + h_a{}^{\omega}g_{\omega\tau}K^{\sigma}{}_{\lambda\sigma} \\ &\quad - h_a{}^{\omega}g_{\omega\lambda}K^{\sigma}{}_{\tau\sigma}\big) \\ &= \left(K^{\mu\lambda}{}_{\sigma}K^{\nu\sigma}{}_{\lambda} - K^{\lambda\mu}{}_{\lambda}K^{\tau\nu}{}_{\tau} - K^{\mu\nu}{}_{\sigma}K^{\tau\sigma}{}_{\tau}\right) \\ &+ \left(-\nabla_\sigma K^{\sigma\nu\mu} + \nabla^{\mu}K^{\sigma\nu}{}_{\sigma} - g^{\mu\nu}\nabla_\tau K^{\sigma\tau}{}_{\sigma}\right) \\ &+ \big(-K^{\mu\lambda}{}_{\sigma}K^{\nu\sigma}{}_{\lambda} - K^{\nu\lambda\sigma}K_{\sigma\lambda}{}^{\mu} + K^{\mu\nu}{}_{\sigma}K^{\tau\sigma}{}_{\tau} \\ &\quad - K^{\lambda\nu\mu}K^{\sigma}{}_{\lambda\sigma} + K^{\lambda\mu}{}_{\lambda}K^{\tau\nu}{}_{\tau} - K_\lambda{}^{\lambda\mu}K^{\tau\nu}{}_{\tau}\big) \\ &+ \frac{1}{4}g^{\mu\nu}\big(K^{\tau\lambda\sigma}K_{\sigma\lambda\tau} + K^{\tau\lambda\sigma}K_{\sigma\lambda\tau} + K_\tau{}^{\tau\lambda}K^{\sigma}{}_{\lambda\sigma} \\ &\quad - K^{\tau\lambda}{}_{\tau}K^{\sigma}{}_{\lambda\sigma} - K^{\tau\lambda}{}_{\tau}K^{\sigma}{}_{\lambda\sigma} + K_\lambda{}^{\lambda\tau}K^{\sigma}{}_{\tau\sigma}\big). \end{aligned}$$

After cancelling terms and using $K_\lambda{}^{\lambda\mu} = 0$ (a consequence of antisymmetry), we obtain the same terms as on the right-hand side. □

A.3 Conservation Law for Gravitational Stress-Energy Tensor (Theorem 5.9)

Proof of Theorem 5.9. Our goal is to prove $f^b + \nabla_\nu t_S^{b\nu} = 0$, where $f^b = T^{\mu b\nu}T_{\mu\nu}$.

Using $\nabla_\nu \overline{T}^{b\nu} = 0$ and $\nabla_\nu T^{\mu\nu} = 0$, we have

$$\begin{aligned} 0 &= \nabla_\nu \overline{T}^{b\nu} \\ &= \nabla_\nu\left(t_S^{b\nu} + T^{b\nu}\right) \\ &= \nabla_\nu t_S^{b\nu} + \nabla_\nu\left(h^b{}_{\mu}T^{\mu\nu}\right) \\ &= \nabla_\nu t_S^{b\nu} + K^b{}_{\mu\nu}T^{\mu\nu} + h^b{}_{\nu}\nabla_\nu T^{\mu\nu} \\ &= \nabla_\nu t_S^{b\nu} + K^b{}_{\mu\nu}T^{\mu\nu}. \end{aligned}$$

Meanwhile, (27) gives

$$\begin{aligned} f^b &= T^{\mu b\nu}T_{\mu\nu} \\ &= \left(K^{\mu\nu b} - K^{\mu b\nu}\right)T_{\mu\nu} \\ &= -K^{\mu b\nu}T_{\mu\nu} \\ &= K^{b\mu\nu}T_{\mu\nu} = K^b{}_{\mu\nu}T^{\mu\nu} \end{aligned}$$

using the anti-symmetry of K and the symmetry of $T^{\mu\nu}$. □