Pseudocharacters of Homomorphisms into Classical Groups

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Abstract

A GL_d-pseudocharacter is a function from a group Γ to a ring k satisfying polynomial relations that make it "look like" the character of a representation. When k is an algebraically closed field of characteristic 0, Taylor proved that GL_d-pseudocharacters of Γ are the same as degree-d characters of Γ with values in k, hence are in bijection with equivalence classes of semisimple representations $\Gamma \to GL_d(k)$. Recently, V. Lafforgue generalized this result by showing that, for any connected reductive group H over an algebraically closed field k of characteristic 0 and for any group Γ, there exists an infinite collection of functions and relations which are naturally in bijection with $H(k)$-conjugacy classes of semisimple homomorphisms $\Gamma \to H(k)$. In this paper, we reformulate Lafforgue's result in terms of a new algebraic object called an FFG-algebra. We then define generating sets and generating relations for these objects and show that, for all H as above, the corresponding FFG-algebra is finitely presented up to radical. Hence one can always define H-pseudocharacters consisting of finitely many functions satisfying finitely many relations. Next, we use invariant theory to give explicit finite presentations up to radical of the FFG-algebras for (general) orthogonal groups, (general) symplectic groups, and special orthogonal groups. Finally, we use our pseudocharacters to answer questions about conjugacy vs. element-conjugacy of homomorphisms, following Larsen.

Introduction

Pseudocharacters were originally introduced for GL_2 by Wiles [Wil] and generalized to GL_n by Taylor [Tay]. Taylor's result on GL_n-pseudocharacters is as follows. Let Γ be a group and k be a commutative ring with identity. Define a *GL_n-pseudocharacter of Γ over k* to be a set map $T : \Gamma \to k$ such that

- $T(1) = n$
- For all $\gamma_1, \gamma_2 \in \Gamma$, $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$
- For all $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$,
$$\sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) T_\sigma(\gamma_1, \dots, \gamma_{n+1}) = 0, \tag{1}$$
where S_{n+1} is the symmetric group on $n+1$ letters, $\operatorname{sgn}(\sigma)$ is the permutation sign of σ, and T_σ is defined by
$$T_\sigma(\gamma_1, \dots, \gamma_{n+1}) = T(\gamma_{i_1^{(1)}} \cdots \gamma_{i_{r_1}^{(1)}}) \cdots T(\gamma_{i_1^{(s)}} \cdots \gamma_{i_{r_s}^{(s)}})$$
where σ has cycle decomposition $(i_1^{(1)} \dots i_{r_1}^{(1)}) \cdots (i_1^{(s)} \dots i_{r_s}^{(s)})$.

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If T is a GL_n-pseudocharacter, then define the kernel of T by

$$\ker(T) = \{\eta \in \Gamma \mid T(\gamma\eta) = T(\gamma) \text{ for all } \gamma \in \Gamma\}.$$

Then:

Theorem 1 ([Tay, Theorem 1]). *1. Let $\rho : \Gamma \to GL_n(k)$ be a representation. Then $\operatorname{tr}(\rho)$ is a GL_n-pseudocharacter.*

2. Suppose k is a field of characteristic 0, and let $\rho : \Gamma \to GL_n(k)$ be a representation. Then $\ker(\operatorname{tr}(\rho)) = \ker(\rho^{ss})$, where ρ^{ss} denotes the semisimplification of ρ.

3. Suppose k is an algebraically closed field of characteristic 0. Let $T : \Gamma \to k$ be a GL_n-pseudocharacter. Then there is a semisimple representation $\rho : \Gamma \to GL_n(k)$ such that $\operatorname{tr}(\rho) = T$, unique up to conjugation.

4. If Γ and k are taken to be topological, then the above statements hold in topological/continuous form.

Taylor used GL_n-pseudocharacters to construct Galois representations having certain properties [Tay, §2].

Recently, V. Lafforgue formulated an analogue of GL_n-pseudocharacters that work with GL_n replaced by any reductive group under conjugation by its identity component. However, instead of consisting of one function $T : \Gamma \to k$ satisfying a finite number of relations, these "pseudocharacters" consist of an infinite sequence of algebra morphisms satisfying certain properties. These sequences of morphisms are essentially equivalent to specifying an infinite number of functions $\Gamma^m \to k$, with m ranging over all natural numbers, satisfying an infinite number of relations. To state Lafforgue's theorem, we adopt the convention that reductive groups are not necessarily connected, with H^0 denoting the identity component of H.

Theorem 2 ([Laf, Proposition 11.7]). *Let Γ be a topological group, k be a topological field of characteristic 0 such that $\overline{k}$ has a topology extending the topology on k, and H be a reductive group over k such that H^0 is split over k.*

For $n \in \mathbb{N}$, let $k[H^n]^{\mathrm{Ad}H^0}$ denote the k-algebra of regular functions on H^n that are invariant under the action of H^0 on H^n by diagonal conjugation, and let $C(\Gamma^n, k)$ denote the k-algebra of continuous set maps $\Gamma^n \to k$.

Assume that we have for any $n \in \mathbb{N}$ a k-algebra morphism

$$\Xi_n : k[H^n]^{\mathrm{Ad}H^0} \to C(\Gamma^n, k)$$

such that

(a) For any $m, n \in \mathbb{N}$, set map $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, $f \in k[H^m]^{\mathrm{Ad}H^0}$, and $\gamma_1, \dots, \gamma_n \in \Gamma$,

$$\Xi_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Xi_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}),$$

where $f^\zeta \in k[H^n]^{\mathrm{Ad}H^0}$ is defined by

$$f^\zeta(A_1, \dots, A_n) = f(A_{\zeta(1)}, \dots, A_{\zeta(m)})$$

(b) For any $n \in \mathbb{N}$, $f \in k[H^n]^{\mathrm{Ad}H^0}$, and $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$,

$$\Xi_{n+1}(\widehat{f})(\gamma_1, \dots, \gamma_{n+1}) = \Xi_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n\gamma_{n+1}),$$

where $\widehat{f} \in k[H^{n+1}]^{\mathrm{Ad}H^0}$ is defined by

$$\widehat{f}(A_1, \dots, A_{n+1}) = f(A_1, \dots, A_{n-1}, A_nA_{n+1}).$$

Then there exists a continuous group homomorphism $\rho : \Gamma \to H(k')$ for some finite extension k' of k (with $H(k')$ inheriting its topology from k'), such that the Zariski closure of $\mathrm{Im}(\rho)$ is a reductive subgroup of $H(k')$, and such that for any $n \in \mathbb{N}$, $f \in k[H^n]^{\mathrm{Ad}H^0}$, and $\gamma_1, \ldots, \gamma_n \in \Gamma^n$, we have

$$f(\rho(\gamma_1), \ldots, \rho(\gamma_n)) = \Xi_n(f)(\gamma_1, \ldots, \gamma_n).$$

Moreover, ρ is unique up to conjugation by $H^0(\overline{k})$.

Remark 1. Lafforgue's original statement requires Γ to be profinite and k to be a finite extension of $\mathbb{Q}_l$ for some l, with the standard topologies. These conditions are not used in the proof, so we omit them.

Lafforgue also shows how to derive Taylor's result from the above theorem [Laf, Remark 11.8], using results of Procesi [Pro2] that state that the trace function "generates" all of the algebras $k[GL_n^m]^{\mathrm{Ad}GL_n}$ and that explicitly describe all of the relations between these trace functions.

Outline

In Section 1, we reformulate Lafforgue's result in terms of a new algebraic structure called an FFG-algebra. Collections of morphisms Ξ_n as above are recast as morphisms between certain FFG-algebras. We then use the finiteness theorems of classical invariant theory and facts about reductive groups to show that, for any connected reductive group H defined over a field of characteristic 0, the FFG-algebra derived from the invariants of H is finitely presented up to radical. Hence it is always possible to define H-pseudocharacters consisting of finitely many functions $\Gamma^m \to k$ satisfying finitely many relations.

In Section 2, we use invariant theoretic-results of Procesi and others [Pro2, ATZ, Rog] to give explicit finite presentations up to radical of the FFG-algebras corresponding to the general and ordinary orthogonal groups GO_n and O_n, the general and ordinary symplectic groups GSp_{2n} and Sp_{2n}, and the special orthogonal group SO_n. By extension, we define explicit pseudocharacters for these groups.

Finally, in Section 3, we use our pseudocharacters to investigate the problem of conjugacy vs. element-conjugacy for semisimple homomorphisms $\Gamma \to H(k)$, where H is a linear algebraic group for which one can define pseudocharacters and k is an algebraically closed field of characteristic 0. We formulate a general condition in terms of FFG-algebras under which element-conjugacy implies conjugacy. We then use our explicit pseudocharacters for $GO_n(k)$, $O_n(k)$, $GSp_{2n}(k)$, $Sp_{2n}(k)$, $SO_{2n+1}(k)$ and $SO_4(k)$ to prove that for any group Γ, element-conjugate semisimple homomorphisms from Γ to one of those groups are automatically conjugate. Previous results of this form were only known for $O_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$, $SO_{2n+1}(\mathbb{C})$, and $SO_4(\mathbb{C})$, and only for compact Γ. We also give a counterexample to the corresponding claim for $SO_{2n}(k)$ ($n \geq 3$) that is simpler than the one in [Lar1, Proposition 3.8], and which extends that result to $SO_6(k)$.

1 General Results on Pseudocharacters

1.1 F-, FFS-, and FFG-algebras.

We begin by defining new algebraic objects which we call F-, FFS-, and FFG-algebras, modeled after the FI-modules defined in [CEF]. For every nonempty finite set I, let $\mathrm{FS}(I)$ (resp. $\mathrm{FG}(I)$) denote the free semigroup (resp. group) generated by I. We denote by F, FFS, and FFG the categories whose objects are finite sets and whose morphisms $I \to J$ are: for F, set functions $I \to J$; for FFS, semigroup homomorphisms $\mathrm{FS}(I) \to \mathrm{FS}(J)$; and for FFG, group homomorphisms $\mathrm{FG}(I) \to \mathrm{FG}(J)$.

The following lemma is easy.

Lemma 3. *The category FFS is generated by the following two types of morphisms:*

- *morphisms* $\mathrm{FS}(I) \to \mathrm{FS}(J)$ *that sends generators to generators, i.e., those induced by maps between finite sets $I \to J$*

- *morphisms*

$$\mathrm{FS}(\{x_1,\ldots,x_n\}) \to \mathrm{FS}(\{y_1,\ldots,y_{n+1}\}), \qquad x_i \mapsto y_i (i<n), x_n \mapsto y_n y_{n+1}.$$

The category FFG is generated by the above two types of morphisms (with FS *replaced by* FG*) together with:*

- *morphisms*

$$\mathrm{FG}(\{x_1,\ldots,x_n\}) \to \mathrm{FG}(\{y_1,\ldots,y_n\}), \qquad x_i \mapsto y_i (i<n), x_n \mapsto y_n^{-1}.$$

Definition 1. Fix a commutative ring k. An *F-algebra* (resp. *FFS-algebra, FFG-algebra*) is a covariant functor from F (resp. FFS, FFG) to the category of k-algebras. Morphisms between F-algebras (resp. FFS-algebras, FFG-algebras) are natural transformations of functors.

If $A^\bullet$ is an F-algebra (resp. FFS-algebra, FFG-algebra) and I is a finite set, we use A^I to denote the k-algebra corresponding to I under $A^\bullet$, and similarly for morphisms $\Theta^\bullet : A^\bullet \to B^\bullet$. For $n \in \mathbb{N}$, we use A^n to denote $A^{\{1,\ldots,n\}}$. If $\phi : I \to J$ (resp. $\mathrm{FS}(I) \to \mathrm{FS}(J)$, $\mathrm{FG}(I) \to \mathrm{FG}(J)$) is a morphism, then we use A^ϕ to denote the corresponding k-algebra morphism $A^I \to A^J$.

We can define kernels, subobjects, quotients, and tensor products over k in the category of F-algebras (resp. FFS-algebras, FFG-algebras) by using the analogous constructions in the category of k-algebras, applying those constructions to each k-algebra in the image of an F-algebra. We say that a morphism $\Theta^\bullet$ is surjective if each Θ^I is surjective.

Remark 2. Any FFG-algebra is naturally an FFS-algebra, and any FFS-algebra is naturally an F-algebra, due to the clear functors $\mathrm{F} \to \mathrm{FFS} \to \mathrm{FFG}$. A morphism of FFG-algebras is also a morphism of FFS-algebras, and a morphism of FFS-algebras is also a morphism of F-algebras.

Example 1. Let Γ be a group and R be a k-algebra. We define an FFG-algebra $\mathrm{Map}(\Gamma^\bullet, R)$ as follows. To the finite set I, we associate $\mathrm{Map}(\Gamma^I, R)$, the k-algebra of all set maps $\Gamma^I \to R$. Next, recall that for any finite set I, $\Gamma^I = \mathrm{Hom}(\mathrm{FG}(I), \Gamma)$. Thus for any group homomorphism $\phi : \mathrm{FG}(I) \to \mathrm{FG}(J)$, we have a natural set map $\Gamma^J \to \Gamma^I$, which induces a k-algebra morphism $\mathrm{Map}(\Gamma^I, R) \to \mathrm{Map}(\Gamma^J, R)$; we associate this morphism to ϕ. When Γ and R are topological, we can analogously define an FFG-algebra $C(\Gamma^\bullet, R)$ by restricting to continuous maps.

Example 2. Let V be an affine variety over k, and let H be a group which acts on V. We define the F-algebra $k[V^\bullet]^H$ by the association $I \mapsto k[V^I]^H$, where H acts diagonally on V^I. For any set map $\phi : I \to J$, we get a variety map $V^J \to V^I$ defined over k, and this induces a k-algebra morphism $k[V^I]^H \to k[V^J]^H$, which we associate to ϕ. If V is also an algebraic semigroup (resp. group) whose multiplication is compatible with the action of H, then we can similarly give $k[V^\bullet]^H$ a structure of FFS-algebra (resp. FFG-algebra). Specifically, given a semigroup homomorphism $\phi : \mathrm{FS}(I) \to \mathrm{FS}(J)$ (resp. group homomorphism $\phi : \mathrm{FG}(I) \to \mathrm{FG}(J)$), letting $I = \{x_1,\ldots,x_n\}$ and $J = \{y_1,\ldots,y_m\}$, we define a variety map

$$\begin{aligned} V(\phi) &: V^J \to V^I \\ (A_1,\ldots,A_m) &\mapsto (\phi(x_1)\{y_i \mapsto A_i\},\ldots,\phi(x_n)\{y_i \mapsto A_i\}), \end{aligned}$$

where $\phi(x_j)\{y_i \mapsto A_i\}$ denotes the point of V obtained by substituting each A_i for y_i in $\phi(x_j)$ and multiplying using the semigroup (resp. group) operation. Then $V(\phi)$ induces a k-algebra morphism $k[V^I]^H \to k[V^J]^H$, which we associate to ϕ.

For the remainder of this section, we state definitions and claims for F-algebras, but they easily generalize to FFS- and FFG-algebras.

Definition 2. Let $A^\bullet$ be an F-algebra. The *arity* of an element $f \in A^I$ is $|I|$. We use this term because in our examples, generally elements of A^I are functions of arity $|I|$.

Definition 3. Let $A^\bullet$ be an F-algebra. Given a subset $\Sigma \subset \sqcup_I A^I$, the *F-algebra span of* Σ *in* $A^\bullet$ is defined to be the minimal sub-F-algebra of $A^\bullet$ containing each element of Σ. An F-algebra is *finitely generated* if it equals the span of some finite set.

There is another way to characterize finite generation, in terms of free F-algebras.

Definition 4. Let $m \in \mathbb{N}$. The *free F-algebra of arity* m, denoted $F_{\mathrm{F}}(m)^\bullet$, is defined by

$$F_{\mathrm{F}}(m)^I = k[\{x_\psi \mid \psi \in \mathrm{Hom}_{\mathrm{F}}(\{1,\dots,m\}, I)\}]$$
$$F_{\mathrm{F}}(m)^\phi = (x_\psi \mapsto x_{\phi\circ\psi}).$$

In the case of FFS-algebras (resp. FFG-algebras), we replace $\mathrm{Hom}_{\mathrm{F}}(\{1,\dots,m\}, I)$ with $\mathrm{Hom}_{\mathrm{FFS}}(\mathrm{FS}(\{1,\dots,m\}), \mathrm{FS}(I))$ (resp. $\mathrm{Hom}_{\mathrm{FFG}}(\mathrm{FG}(\{1,\dots,m\}), \mathrm{FG}(I))$).

It is easy to see that $F_{\mathrm{F}}(m)^\bullet$ has the universal property: if $A^\bullet$ is an F-algebra and $a \in A^m$, then there is a unique F-algebra morphism $F_{\mathrm{F}}(m)^\bullet \to A^\bullet$ mapping $x_{id_{\{1,\dots,m\}}}$ to a. Furthermore, the image of this morphism is precisely the span of a in $A^\bullet$. Thus:

Proposition 4. *An F-algebra* $A^\bullet$ *is finitely generated iff it admits a surjective morphism* $\bigotimes_i F_{\mathrm{F}}(m_i) \to A^\bullet$ *for some finite sequence of integers* (m_i).

Definition 5. Let $A^\bullet$ be an F-algebra. An *F-ideal of* $A^\bullet$ is an association $\mathfrak{a}^\bullet$ taking each finite set I to an ideal $\mathfrak{a}^I$ of A^I, such that for all morphisms $\phi \in \mathrm{Hom}_{\mathrm{F}}(I, J)$, we have $A^\phi(\mathfrak{a}^I) \subset \mathfrak{a}^J$. Given a morphism of F-algebras $\Theta^\bullet : A^\bullet \to B^\bullet$, we define the *kernel* of $\Theta^\bullet$ to be the association $\ker(\Theta^\bullet)$ taking each finite set I to the ideal $\ker(\Theta^I : A^I \to B^I)$ of A^I. We define the *radical* of an F-ideal $\mathfrak{a}^\bullet$ to be the association $\sqrt{\mathfrak{a}^\bullet} : I \mapsto \sqrt{\mathfrak{a}^I}$, where the radical of $\mathfrak{a}^I$ is taken in A^I. Easily kernels and radicals are F-ideals.

Definition 6. Let $A^\bullet$ be an F-algebra. Given a subset $\Sigma \subset \sqcup_I A^I$, we define the *F-ideal generated by* Σ to be the minimal F-ideal of $A^\bullet$ containing each element of Σ. We define an F-ideal to be *finitely generated* if it is generated by some finite set. We call an F-algebra $A^\bullet$ *finitely presented* if it admits a surjective morphism $\pi^\bullet : \bigotimes_i F_{\mathrm{F}}(m_i) \to A^\bullet$ for some finite sequence of integers (m_i) such that $\ker(\pi^\bullet)$ is finitely generated. We call $A^\bullet$ *finitely presented up to radical* if $\ker(\pi^\bullet) = \sqrt{I^\bullet}$ for some finitely generated F-ideal $I^\bullet$.

1.2 Pseudocharacters from Lafforgue's Result.

Let H be a reductive group defined over a topological field k, and let Γ be a topological group. Let H^0 denote the identity component of H (in the Zariski topology). For any finite set I, let $\mathrm{Ad}H^0$ denote the diagonal conjugation action of H^0 on H^I, and let $k[H^\bullet]^{\mathrm{Ad}H^0}$ denote the FFG-algebra in Example 2 corresponding to this action. Call a homomorphism $\rho : \Gamma \to H(k)$ *semisimple* if the Zariski closure of $\mathrm{Im}(\sigma)$ in $H(k)$ is reductive. Then from V. Lafforgue's result, we derive the following generalization of Taylor's pseudocharacters.

Theorem 5. *(1) Let* $\rho : \Gamma \to H(k)$ *be a continuous (with the* k*-topology on* $H(k)$*) homomorphism. Then we have an* FFG*-algebra morphism*

$$\Theta^\bullet : k[H^\bullet]^{\mathrm{Ad}H^0} \to C(\Gamma^\bullet, k)$$

given by

$$\Theta^n(f)(\gamma_1,\dots,\gamma_n) = f(\rho(\gamma_1),\dots,\rho(\gamma_n)).$$

(2) Conversely, suppose k *is a field of characteristic 0, and let* $\overline{k}$ *have a topology extending the topology on* k*. Let*

$$\Theta^\bullet : k[H^\bullet]^{\mathrm{Ad}H^0} \to C(\Gamma^\bullet, k)$$

be an FFS*-algebra morphism. Then there is a finite extension k' of k and a continuous semisimple homomorphism $\rho : \Gamma \to H(k')$ such that*

$$\Theta^n(f)(\gamma_1, \dots, \gamma_n) = f(\rho(\gamma_1), \dots, \rho(\gamma_n)).$$

Moreover, ρ is unique up to conjugation by $H^0(\overline{k})$. Note that by (1), $\Theta^\bullet$ is also an FFG*-algebra morphism.*

(3) Suppose k is a field of characteristic 0 and H is connected. Let $\rho : \Gamma \to H(k)$ be a semisimple homomorphism. Then

$$\ker(\rho) = \Big\{\eta \in \Gamma \mid \textit{for all } n \in \mathbb{N},\ f \in k[H^n]^{\mathrm{Ad}H},\ 1 \le i \le n, \textit{ and } \gamma_1, \dots, \gamma_n \in \Gamma,$$
$$f(\rho(\gamma_1), \dots, \rho(\eta\gamma_i), \dots, \rho(\gamma_n)) = f(\rho(\gamma_1), \dots, \rho(\gamma_n))\Big\}.$$

Remark 3. When Γ and k are not topological, we can give them the discrete topology and then apply the above theorem, giving an analogous result with $C(\Gamma^\bullet, k)$ replaced by $\mathrm{Map}(\Gamma^\bullet, k)$.

Proof. (1) This is easily checked.

(2) It suffices to prove the claim with k replaced by a finite extension, so without loss of generality H^0 is split over k. Then this follows from Lafforgue's result [Laf, Proposition 11.7], as stated in Theorem 2 above, noting that the conditions on the Ξ_n in that result are precisely the FFS-algebra morphism conditions for the generators in Lemma 3.

(3) Let Δ be the right-hand set. Easily Δ is a normal subgroup of Γ, so we can form the quotient $\overline{\Gamma} = \Gamma/\Delta$. Let $\Theta^\bullet$ be the FFG-algebra morphism corresponding to ρ in (1). Then by definition of Δ, $\Theta^\bullet$ restricts to give a well-defined FFG-algebra morphism

$$\overline{\Theta}^\bullet : k[H^\bullet]^{\mathrm{Ad}H} \to \mathrm{Map}(\overline{\Gamma}^\bullet, k).$$

Hence by (2), we have a corresponding semisimple homomorphism $\overline{\psi} : \overline{\Gamma} \to H(\overline{k})$. Composing with the quotient map gives a continuous semisimple homomorphism $\psi : \Gamma \to H(\overline{k})$. From the definition of Δ, $\overline{\psi}$ has trivial kernel, so $\ker(\psi) = \Delta$. Finally, by uniqueness in (2), $\psi = \rho$.

□

In Remark 5 below, we prove that this result still holds if we replace H^0 by an arbitrary connected reductive group K such that $H^0 \subset K$ and K normalizes H, i.e., $K \subset \mathrm{N}_G(H)$ for some ambient reductive group G containing H and K.

1.3 Explicit Descriptions of Pseudocharacters.

Let k be a field of characteristic 0, and let H be a reductive group over k. We now show that the FFG-algebra $k[H^\bullet]^{\mathrm{Ad}H^0}$ is finitely presented up to radical. More generally, $k[H^\bullet]^{\mathrm{Ad}K}$ is finitely presented up to radical whenever K is a reductive group normalizing H (Remark 5).

As a consequence, when H is connected, i.e., $H = H^0$, it is always possible to define pseudocharacters for H very explicitly. Indeed, if $k[H^\bullet]^{\mathrm{Ad}H}$ has a finite presentation up to radical with generators $f_1, \dots, f_a$ of arities $n_1, \dots, n_a$ and generating relations $R_1, \dots, R_b$, then to specify an FFG-algebra morphism $k[H^\bullet]^{\mathrm{Ad}H} \to C(\Gamma^\bullet, k)$, it is equivalent to specify continuous set maps $F_1 : \Gamma^{n_1} \to k, \dots, F_a : \Gamma^{n_a} \to k$ satisfying the relations $R_1, \dots, R_b$. Hence we can define explicit pseudocharacters for H by finding a finite presentation up to radical of $k[H^\bullet]^{\mathrm{Ad}H}$. This technique was first demonstrated in [Laf, Remark 11.8], in which V. Lafforgue implicitly gives a finite presentation up to radical of $k[GL_n^\bullet]^{\mathrm{Ad}GL_n}$ and explains how it implies Taylor's original result on GL_n-pseudocharacters. We further illustrate this technique with examples in Section 2 below.

Lemma 6. *Let H^0 act linearly on a finite-dimensional k-vector space V. Then the* F*-algebra $k[V^\bullet]^{H^0}$ is finitely presented.*

Proof. Finite generation follows from a strong form of the first fundamental theorem of invariant theory, which is classical; see, e.g., [PV, Corollary on p. 253]. Finite presentation follows from a strong form of the second fundamental theorem of invariant theory proven by Schwarz [Sch, Theorem 2.5(2)][1]. □

From this result, we easily deduce that $k[H^\bullet]^{\mathrm{Ad}H^0}$ is finitely generated (see the proof of Theorem 9). It remains to prove a finiteness result for the relations between the generators. In the classical case of a reductive group acting on a single vector space, finite generation of relations follows immediately from the Noetherian property of finitely generated k-algebras. However, such a Noetherian property does not hold for finitely generated F-algebras. Nagel and Römer [NR, Proposition 4.8] show that the free FI-algebra $F_{\mathrm{FI}}(m)$ is not Noetherian for all $m \geq 2$, where FI denotes the category of finite sets with injective maps, and their proof easily generalizes to F-, FFS-, and FFG-algebras.

Instead, we reason about $k[H^\bullet]^{\mathrm{Ad}H^0}$ in particular, using properties of reductive groups. We start with two lemmas.

Lemma 7. *Assume k is algebraically closed. Then for all $d \in \mathbb{N}$, there exists $q_d \in \mathbb{N}$ such that any algebraic subgroup $G \subset GL_d(k)$ is generated by at most q_d elements as an algebraic group, i.e., there exist $g_1, \ldots g_{q_d} \in G$ such that $G = \overline{\langle g_1, \ldots, g_{q_d} \rangle}$, where the closure is taken in the Zariski topology.*

Proof. By a result of Mostow, G is the semidirect product of a reductive group and a unipotent group, namely, a Levi subgroup and the unipotent radical of G [Hoc, Theorem VIII.4.3]. Hence it suffices to prove the lemma when G is reductive or unipotent.

First, consider the case that G is reductive. By [Vin, Propositions 2 and 7], we reduce to the case that G is finite. By Jordan's theorem on finite linear groups, we reduce to the case that G is finite and abelian. Then the faithful representation of G on k^d decomposes as a sum of 1-dimensional representations, so after conjugating, we can assume G is a subgroup of the diagonal group $(k^\times)^d$. Since G is finite, it is isomorphic to a subgroup of $\boldsymbol{\mu}^d \cong (\mathbb{Q}/\mathbb{Z})^d$, hence to a subgroup of $(\mathbb{Z}/n\mathbb{Z})^d$ for some n. Any such subgroup is generated by at most d elements.

Now consider the case that G is unipotent. Then G has a composition series (as an algebraic group) $G = U_1 \supset \cdots \supset U_s = \{e\}$ in which all of the quotients U_i/U_{i+1} are isomorphic to $\mathbf{G}_a$, the additive group of k. Any non-identity element of $\mathbf{G}_a(k)$ topologically generates $\mathbf{G}_a$, since such an element generates an infinite subgroup of $\mathbf{G}_a(k)$ and all proper Zariski closed subgroups of $\mathbf{G}_a(k)$ are finite. Thus G is topologically generated by a set containing one element from each $U_i \setminus U_{i+1}$, which has size $s - 1 = \dim G$. But $\dim G$ is bounded by $\dim GL_d$. □

Let d be such that H is an affine subgroup variety of GL_d. Let $GL_d^n//\mathrm{Ad}H^0$ denote the variety $\mathrm{Spec}(k[GL_d^n]^{\mathrm{Ad}H^0})$, and similarly for $H^n//\mathrm{Ad}H^0$. Let $\pi_{H^0}^n : GL_d^n \to (GL_d^n//\mathrm{Ad}H^0)$ be the natural projection. For any group homomorphism $\phi : \mathrm{FG}(\{x_1, \ldots, x_m\}) \to \mathrm{FG}(\{y_1, \ldots, y_n\})$, we get a map $V(\phi) : (GL_d^n//\mathrm{Ad}H^0) \to (GL_d^m//\mathrm{Ad}H^0)$ from the k-algebra morphism $(k[GL_d^\bullet]^{\mathrm{Ad}H^0})^\phi$. Concretely, for $A_1, \ldots, A_n \in GL_d(k)$, $V(\phi)$ sends $\pi_{H^0}^n(A_1, \ldots, A_n)$ to $\pi_{H^0}^m(\phi(x_1)\{y_i \mapsto A_i\}, \ldots, \phi(x_m)\{y_i \mapsto A_i\})$, where $\phi(x_j)\{y_i \mapsto A_i\}$ denotes the element of $GL_d(k)$ obtained by substituting each A_i for y_i in $\phi(x_j)$.

Lemma 8. *Assume k is algebraically closed. Let d be such that H is an affine subgroup variety of GL_d, and let q_d be the constant from Lemma 7 for GL_d. Suppose $x \in GL_d^n//\mathrm{Ad}H^0$ is such that for all group homomorphisms $\phi : \mathrm{FG}(\{x_1, \ldots, x_{q_d}\}) \to \mathrm{FG}(\{y_1, \ldots, y_n\})$, $V(\phi)(x) \in (H^{q_d}//\mathrm{Ad}H^0) \subset (GL_d^{q_d}//\mathrm{Ad}H^0)$. Then $x \in H^n//\mathrm{Ad}H^0$.*

Proof. It is a well-known property of $\pi_{H^0}^n$ that the preimage of any element of $GL_d^n//\mathrm{Ad}H^0$ in GL_d^n contains a unique closed orbit under conjugation by H^0. Thus we can find a preimage $(A_1, \ldots, A_n) \in GL_d^n(k)$ of x whose orbit under conjugation by H^0 is closed in GL_d^n. By the previous lemma, we can find $B_1, \ldots, B_{q_d} \in GL_d(k)$ such that $\overline{\langle A_1, \ldots, A_n \rangle} = \overline{\langle B_1, \ldots, B_{q_d} \rangle}$. Each B_i is in the closure of $\langle A_1, \ldots, A_n \rangle$, so $\pi_{H^0}^{q_d}(B_1, \ldots, B_{q_d})$ is in the closure of $\{\pi_{H^0}^{q_d}(C_1, \ldots, C_{q_d}) \mid C_1, \ldots, C_{q_d} \in \langle A_1, \ldots, A_n \rangle\}$. Then $\pi_{H^0}^{q_d}(B_1, \ldots, B_{q_d}) \in H^{q_d}//\mathrm{Ad}H^0$ because all $\pi_{H^0}^{q_d}(C_1, \ldots, C_{q_d}) \in H^{q_d}//\mathrm{Ad}H^0$ by assumption.

[1] I thank C. Procesi and G. Schwarz for their assistance in locating this result.

Next, the orbit of $(B_1, \ldots, B_{q_d})$ is closed because the orbit of $(A_1, \ldots, A_n)$ is closed: indeed, by [Ric1], the orbit of a tuple under conjugation by H^0 is closed iff its stabilizer in H^0 is reductive, and the stabilizer of a tuple only depends on the algebraic subgroup it generates in GL_d. This closed orbit must coincide with the unique closed orbit in the preimage of $\pi_{H^0}^{q_d}(B_1, \ldots, B_{q_d})$ in H^{q_d}, so all $B_i \in H(k)$. Hence all $A_j \in H(k)$, proving the lemma. □

Theorem 9. *The* FFG*-algebra* $k[H^\bullet]^{\mathrm{Ad}H^0}$ *is finitely presented up to radical.*

Proof. Let d be such that H is an affine subgroup variety of GL_d over k. Then we have closed embeddings $H \hookrightarrow GL_d \hookrightarrow M_d \times \mathbb{A}^1$, where M_d is the variety of $d \times d$ matrices, $\mathbb{A}_1$ is one-dimensional affine space, and the last embedding is given by $A \mapsto (A, \det(A)^{-1})$. These embeddings are compatible with the conjugation action by H. Then we get an F-algebra morphism $\pi_1^\bullet : k[(M_d \times \mathbb{A}^1)^\bullet]^{\mathrm{Ad}H^0} \to k[GL_d^\bullet]^{\mathrm{Ad}H^0}$ and an FFG-algebra morphism $\pi_2^\bullet : k[GL_d^\bullet]^{\mathrm{Ad}H^0} \to k[H^\bullet]^{\mathrm{Ad}H^0}$. It is a standard fact (see, e.g., [PV, p. 188]) that each π_1^I and π_2^I are surjective, so $\pi_1^\bullet$ and $\pi_2^\bullet$ are surjective.

By Lemma 6, $k[(M_d \times \mathbb{A}^1)^\bullet]^{\mathrm{Ad}H^0}$ is finitely presented as an F-algebra. Next, $\ker(\pi_1^\bullet)$ is generated as an F-ideal by the relation det(matrix coordinate) $\cdot$ (affine coordinate) $= 1$ in $k[M_d \times \mathbb{A}^1]^{\mathrm{Ad}H^0}$, so $k[GL_d^\bullet]^{\mathrm{Ad}H^0}$ is finitely presented as an F-algebra. From such a finite presentation, together with relations expressing $f(A_1, \ldots, A_{n-1}, A_nA_{n+1})$ and $f(A_1, \ldots, A_{n-1}, A_n^{-1})$ in terms of the F-algebra for each of the finitely many generators f, we get a finite presentation of $k[GL_d^\bullet]^{\mathrm{Ad}H^0}$ as an FFG-algebra.

Now to prove the theorem, it suffices to show that $\ker(\pi_2^\bullet)$ is the radical of a finitely generated FFG-ideal in $k[GL_d^\bullet]^{\mathrm{Ad}H^0}$. This is more difficult to show; while the kernel of the natural map $k[GL_d^\bullet] \to k[H^\bullet]$ is generated by $\ker(k[GL_d] \to k[H])$, the same is not necessarily true once we restrict to the algebras of invariants.

It suffices to show this for $\overline{k}$, so without loss of generality k is algebraically closed. Let q_d be the constant from Lemma 7 for GL_d. Let $I^\bullet$ be the FFG-ideal of $k[GL_d^\bullet]^{\mathrm{Ad}H^0}$ generated by $\ker(\pi_2^{q_d})$. $I^\bullet$ is finitely generated, so it suffices to prove $\ker(\pi_2^\bullet) = \sqrt{I^\bullet}$. That is, we must show $\ker(\pi_2^n) = \sqrt{I^n}$ for all n. By the Nullstellensatz, it is equivalent to show that $\ker(\pi_2^n)$ and I^n define the same subvariety of $\mathrm{Spec}(k[GL_d^n]^{\mathrm{Ad}H^0})$. This follows from Lemma 8. □

Remark 4. The same proof shows that $k[H^\bullet]^{\mathrm{Ad}H^0}$ is finitely presented up to radical as an FFS-algebra.

Remark 5. By modifying the above proofs, we can generalize our main result on pseudocharacters (Theorem 5) to the case when H^0 is replaced by an arbitrary connected reductive group K such that $H^0 \subset K$ and K normalizes H, i.e., $K \subset \mathrm{N}_G(H)$ for some ambient reductive group G containing H and K. Furthermore, the FFG-algebra $k[H^\bullet]^{\mathrm{Ad}K}$ is finitely presented up to radical; in fact, this holds for any reductive group K normalizing H (not necessarily connected). Thus there exist explicit pseudocharacters for semisimple homomorphisms $\Gamma \to H(\overline{k})$ considered up to conjugation by $K(\overline{k})$.

To prove these results, first let K be any reductive group normalizing H (not necessarily connected). Let d be such that H and K are affine subgroup varieties of GL_d. Observe the following modification of Lemma 8: if $x \in GL_d^n//\mathrm{Ad}K$ is such that for all group homomorphisms $\phi : \mathrm{FG}(q_d) \to \mathrm{FG}(n)$, $V(\phi)(x) \in H^{q_d}//\mathrm{Ad}K$, then $x \in H^n//\mathrm{Ad}K$. The proof is the same, noting that $H^{q_d}//\mathrm{Ad}K$ is closed in $GL_d^n//\mathrm{Ad}K$ because K normalizes H.

As in the proof of Theorem 9, it follows that the kernel of the natural surjective FFG-algebra morphism $\pi^\bullet : k[GL_d^\bullet]^{\mathrm{Ad}K} \twoheadrightarrow k[H^\bullet]^{\mathrm{Ad}K}$ is generated by $\ker(\pi^{q_d})$ up to radical. Hence $k[H^\bullet]^{\mathrm{Ad}K}$ is finitely presented up to radical because $k[GL_d^\bullet]^{\mathrm{Ad}K}$ is, noting that the proof of Lemma 6 still holds with K in place of H^0.

Now let K be a connected reductive group such that $H^0 \subset K$ and K normalizes H. It remains to prove Theorem 5 with K in place of H^0. Claim (1) is easily checked, and claim (3) is unchanged. To prove claim (2), let $L = KH$, so that $L^0 = K$, and let $\psi^\bullet : k[L^\bullet]^{\mathrm{Ad}K} \twoheadrightarrow k[H^\bullet]^{\mathrm{Ad}K}$ be the natural map. From an FFG-algebra morphism $\Theta^\bullet : k[H^\bullet]^{\mathrm{Ad}K} \to C(\Gamma^\bullet, k)$, by Theorem 5 applied to L and $\Theta^\bullet \circ \psi^\bullet$, we get a continuous semisimple homomorphism $\rho : \Gamma \to L(k')$ for some finite extension k' of k, unique up to conjugation by $K(\overline{k})$, with K-invariants given by $\Theta^\bullet \circ \psi^\bullet$. In particular, the K-invariants satisfy all relations in $\ker(\psi^{q_d})$. Then as in the proof of Lemma 8, a q_d-tuple $(B_1, \ldots, B_{q_d})$ generating $\mathrm{Im}(\rho)$ in the Zariski topology projects to an element of $H^{q_d}//\mathrm{Ad}K$. Also, since ρ is semisimple, $(B_1, \ldots, B_{q_d})$ is semisimple in the sense of [Ric2], so the K-orbit of $(B_1, \ldots, B_{q_d})$ is closed by [Ric2, Theorem 3.6]. Then $(B_1, \ldots, B_{q_d}) \in H^{q_d}(\overline{k})$ as in the proof of Lemma 8. Thus $\mathrm{Im}(\rho) \subset H(k')$, proving claim (2).

2 Explicit Pseudocharacters for Classical Groups

2.1 (General) Orthogonal Group

We now present new results that establish pseudocharacters for the orthogonal and general orthogonal groups. Let k be a field of characteristic 0.

Let $GO_n(k) = \{A \in M_n(k) \mid \text{for some } \lambda \in k^\times,\ AA^t = \lambda I\}$ be the n-dimensional general orthogonal group. It is a connected reductive algebraic group. Define a function $\lambda : GO_n(k) \to k^\times$ by $AA^t = \lambda(A)I$. Then $\lambda \in k[GO_n]^{\mathrm{Ad}GO_n}$.

Proposition 10. *$k[GO_n^\bullet]^{\mathrm{Ad}GO_n}$ is generated as an FFG-algebra by the arity 1 functions* tr *and* λ.

Proof. Since $GO_n(k) \supset O_n(k)$, $k[M_n^\bullet]^{\mathrm{Ad}GO_n} \subset k[M_n^\bullet]^{\mathrm{Ad}O_n}$. By Procesi's results on the invariants of $O_n(k)$ acting on matrices by conjugation [Pro2, Theorem 7.1], for all m, $k[M_n^m]^{\mathrm{Ad}O_n}$ is generated as a k-algebra by invariants $\mathrm{tr}(M)$, where $M \in \mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$. The $\mathrm{tr}(M)$ are obviously also $GO_n(k)$-invariants, so $k[M_n^m]^{\mathrm{Ad}GO_n}$ has the same generators. Then $k[(M_n \times \mathbb{A}^1)^m]^{\mathrm{Ad}GO_n}$ is generated as a k-algebra by the $\mathrm{tr}(M)$ and by the coordinate functions for the m copies of $\mathbb{A}^1$, which we will denote $\det^{-1}(A_1), \ldots, \det^{-1}(A_m)$.

Next, $k[GO_n^m]^{\mathrm{Ad}GO_n}$ is a quotient of $k[(M_n \times \mathbb{A}^1)^m]^{\mathrm{Ad}GO_n}$ for all m because GO_n is an affine subvariety of GL_n, so $k[GO_n^m]^{\mathrm{Ad}GO_n}$ is also generated by the invariants $\mathrm{tr}(M)$ and $\det^{-1}(A_i)$. Then using the identity $A^t = \lambda(A)A^{-1}$ for $A \in GO_n(k)$, we see that any invariant $\mathrm{tr}(M)$ is in the FFG-algebra generated by tr and λ. Also, using the identity $\det^{-1}(A) = \det(A^{-1})$ and the fact that we can express $\det(A^{-1})$ in terms of $\mathrm{tr}(A^{-1}), \ldots, \mathrm{tr}(A^{-n})$, we see that any invariant $\det^{-1}(A_i)$ is in the FFG-algebra generated by tr. □

The relations between the invariants are more complicated to describe. We first summarize Procesi's result on relations between the generators $\mathrm{tr}(M)$ of $k[M_n^m]^{\mathrm{Ad}GO_n} = k[M_n^m]^{\mathrm{Ad}O_n}$.

Let R be the polynomial ring over k with indeterminates T_M as M varies over $\mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$, except that we make the identifications $T_{MN} = T_{NM}$ and $T_M = T_{M^t}$ for all words M and N, where M^t is defined by reversing the order of letters in M and swapping each $A_i \leftrightarrow A_i^t$. Let $\pi : R \to k[M_n^m]^{\mathrm{Ad}GO_n}$ be the k-algebra homomorphism sending each T_M to $\mathrm{tr}(M)$, which by [Pro2, Theorem 7.1] is surjective.

Given $M_1, M_2, \ldots, M_{n+1} \in \mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$ and an integer $0 \le j \le (n+1)/2$, define $F_{j,n+1}(M_1, M_2, \ldots, M_{n+1}) \in R$ as follows. Let s be given by $n + 1 = 2j + s$. Let S be a set of formal symbols (a, b), where each a and b is one of the formal symbols $u_1, \ldots, u_{n+1}, v_1, \ldots, v_{n+1}$. Let $D^j = \sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) D^j_\sigma$ be the following $(n+1) \times (n+1)$ determinant, as a function of symbols in S:

$$\begin{vmatrix} (u_1, u_{j+s+1}) & \cdots & (u_1, u_{n+1}) & (u_1, v_{j+1}) & \cdots & (u_1, v_{n+1}) \\ \vdots & & \vdots & \vdots & & \vdots \\ (u_{j+s}, u_{j+s+1}) & \cdots & (u_{j+s}, u_{n+1}) & (u_{j+s}, v_{j+1}) & \cdots & (u_{j+s}, v_{n+1}) \\ (v_1, u_{j+s+1}) & \cdots & (v_1, u_{n+1}) & (v_1, v_{j+1}) & \cdots & (v_1, v_{n+1}) \\ \vdots & & \vdots & \vdots & & \vdots \\ (v_j, u_{j+s+1}) & \cdots & (v_j, u_{n+1}) & (v_j, v_{j+1}) & \cdots & (v_j, v_{n+1}) \end{vmatrix}$$

Next, using the formal identities $(a, b) = (b, a)$ and allowing the symbols (a, b) to commute with each other, write each monomial D^j_σ of D^j in the form

$$D^j_\sigma = (w_{i_1^{(1)}}, \bar{w}_{i_2^{(1)}})(w_{i_2^{(1)}}, \bar{w}_{i_3^{(1)}}) \cdots (w_{i_{r_1}^{(1)}}, \bar{w}_{i_1^{(1)}}) \cdot (w_{i_1^{(2)}}, \bar{w}_{i_2^{(2)}})(w_{i_2^{(2)}}, \bar{w}_{i_3^{(2)}}) \cdots (w_{i_{r_2}^{(2)}}, \bar{w}_{i_1^{(2)}}) \cdots$$

where w_a stands for either u_a or v_a and we define $\bar{u}_a = v_a$ and $\bar{v}_a = u_a$. Now define $T^j_\sigma(M_1, \ldots, M_{n+1})$ by

$$T^j_\sigma(M_1, \ldots, M_{n+1}) = T_{N_{i_1^{(1)}} N_{i_2^{(1)}} \ldots N_{i_{r_1}^{(1)}}} T_{N_{i_1^{(2)}} N_{i_2^{(2)}} \ldots N_{i_{r_2}^{(2)}}} \cdots$$

where $N_a = M_a$ or $N_a = M_a^t$, according to the inductively defined rules:

- $N_{i_1^{(k)}} = M_{i_1^{(k)}}$, if $w_{i_1^{(k)}} = v_{i_1^{(k)}}$; else $N_{i_1^{(k)}} = M^t_{i_1^{(k)}}$
- Set $N_{i_{t+1}^{(k)}}$ to be the same type as $N_{i_t^{(k)}}$ (transposed or not transposed) if and only if $w_{i_t^{(k)}}$ and $w_{i_{t+1}^{(k)}}$ stand for instances of the same letter (u or v).

Then

$$F_{j,n+1}(M_1, \ldots, M_{n+1}) = \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) T_\sigma^j(M_1, \ldots, M_{n+1})$$

is the result of replacing each D_σ^j with T_σ^j in D^j. Because $T_{MN} = T_{NM}$ and $T_M = T_{M^t}$ by assumption, the functions T_σ^j are well-defined, hence so is $F_{j,n+1}$. Note that $F_{0,n+1}(M_1, \ldots, M_{n+1})$ reduces to (1), the non-trivial relation for GL_n-pseudocharacters.

Theorem 11 ([Pro2, Theorem 8.4(a)])**.** $\ker(\pi)$ *is the ideal of* R *generated by the* $F_{j,n+1}(M_1, \ldots, M_{n+1})$, $0 \le j \le (n+1)/2$, *as the* M_i *vary over* $\mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$.

Now let $\psi : R \to k[GO_n^m]^{\mathrm{Ad}GO_n}$ be given by

$$\psi : R \xrightarrow{\pi} k[M_n^m]^{\mathrm{Ad}GO_n} \hookrightarrow k[(M_n \times \mathbb{A}^1)^m]^{\mathrm{Ad}GO_n} \twoheadrightarrow k[GO_n^m]^{\mathrm{Ad}GO_n}.$$

Note that ψ is surjective by the proof of Proposition 10. Intuitively, one should expect $\ker(\psi)$ to be $\ker(\pi)$ plus the relations of the form $T_{NN^tP} = \frac{1}{n} T_{NN^t} T_P$, since GO_n is defined by the condition that NN^t is a scalar matrix for all $N \in GO_n(k)$. The next proposition shows that this is indeed the case, at least up to radical.

Proposition 12. $\ker(\psi)$ *is the radical of the ideal generated by* $\ker(\pi)$ *and the relations* $T_{NN^tP} - \frac{1}{n} T_{NN^t} T_P$ *for* $N, P \in \mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$.

Proof. It suffices to show this for $\overline{k}$, so without loss of generality k is algebraically closed. Let $J \subset R$ be the ideal generated by $\ker(\pi)$ and the $T_{NN^tP} - \frac{1}{n} T_{NN^t} T_P$. It suffices to prove that $\pi(\ker(\psi)) = \sqrt{\pi(J)}$. Now $\sqrt{\pi(\ker(\psi))} = \pi(\ker(\psi))$ because $k[M_n^m]^{\mathrm{Ad}GO_n}/\pi(\ker(\psi)) \cong k[GO_n^m]^{\mathrm{Ad}GO_n}$ is reduced, so by the Nullstellensatz, it suffices to prove that $\pi(\ker(\psi))$ and $\pi(J)$ define the same subvariety of $\mathrm{Spec}(k[M_n^m]^{\mathrm{Ad}GO_n})$. Using the $\mathrm{tr}(M)$ as coordinate functions for $\mathrm{Spec}(k[M_n^m]^{\mathrm{Ad}GO_n})$, the subvariety associated to $\pi(\ker(\psi))$ is the set of all points of the form $(\mathrm{tr}(M\{A_i \mapsto B_i\}))_{M \in \mathrm{FS}\{A_1, A_1^t, \ldots, A_m, A_m^t\}}$ for some $B_1, \ldots, B_m \in GO_n(k)$, where $M\{A_i \mapsto B_i\}$ denotes the element of $GO_n(k)$ obtained by substituting each B_i for A_i in M. Meanwhile, the subvariety associated to $\pi(J)$ is the set of all points of the form $(\mathrm{tr}(M\{A_i \mapsto C_i\}))_{M \in \{A_1, A_1^t, \ldots, A_m, A_m^t\}}$ where $C_1, \ldots, C_m \in M_n(k)$ are such that $\mathrm{tr}(NN^tP) = \frac{1}{n} \mathrm{tr}(NN^t)\mathrm{tr}(P)$ whenever N and P are semigroup words in the C_i and C_i^t. The following lemma shows that these two subvarieties are equal, proving the proposition. $\square$

Lemma 13. *Let* $C_1, \ldots, C_m \in M_n(\overline{k})$ *be such that* $\mathrm{tr}(NN^tP) = \frac{1}{n} \mathrm{tr}(NN^t)\mathrm{tr}(P)$ *whenever* N *and* P *are semigroup words in the* C_i *and* C_i^t. *Then there exist* $B_1, \ldots, B_m \in GO_n(\overline{k})$ *such that for all* $M \in \mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$, $\mathrm{tr}(M\{A_i \mapsto B_i\}) = \mathrm{tr}(M\{A_i \mapsto C_i\})$.

Proof. Let (V, B) be the bilinear space with $V \cong \overline{k}^n$ and B the standard nondegenerate symmetric bilinear form, i.e., the dot product. Let A be the noncommutative $\overline{k}$-algebra

$$A = \overline{k}[C_1, \ldots, C_m, C_1^t, \ldots, C_m^t] \subset M_n(\overline{k}),$$

which has the natural involution $(-)^t$. Then the natural representation $\rho : A \hookrightarrow M_n(\overline{k}) \cong \mathrm{End}(V, B)$ is orthogonal, i.e., it preserves involutions.

Then by [Pro2, Theorem 15.2(b)(c)] and the fact that all nondegenerate bilinear forms on V are equivalent, there exists a semisimple orthogonal representation $\rho^{ss} : A \to \mathrm{End}(V)$ such that

$\mathrm{tr}(\rho) = \mathrm{tr}(\rho^{ss})$. Thus setting $B_i = \rho^{ss}(C_i)$, we will be done once we prove that $B_i \in GO_n(\overline{k})$. Now for any $D \in A$, we have

$$\begin{aligned}\mathrm{tr}((B_iB_i^t - \frac{1}{n}\mathrm{tr}(B_iB_i^t)I)\rho^{ss}(D)) &= \mathrm{tr}(\rho^{ss}((C_iC_i^t - \frac{1}{n}\mathrm{tr}(C_iC_i^t)I)D)) \\ &= \mathrm{tr}(\rho((C_iC_i^t - \frac{1}{n}\mathrm{tr}(C_iC_i^t)I)D)) \\ &= \mathrm{tr}((C_iC_i^t)D) - \frac{1}{n}\mathrm{tr}(C_iC_i^t)\mathrm{tr}(D) \\ &= 0\end{aligned}$$

by assumption. Since ρ^{ss} is semisimple, tr is a nondegenerate bilinear form on $\mathrm{Im}(\rho^{ss})$, so this shows that $B_iB_i^t - \frac{1}{n}\mathrm{tr}(B_iB_i^t)I = 0$, proving the lemma. □

Next, let S be the polynomial ring over k with indeterminates:

- U_Q for $Q \in \mathrm{FG}(\{A_1, \ldots, A_m\})$, with the identifications $U_1 = n$ and $U_{QR} = U_{RQ}$ for all words Q, R
- l_Q for $Q \in \mathrm{FG}(\{A_1, \ldots, A_m\})$, with the identifications $l_1 = 1$ and $l_{QR} = l_Q l_R$ for all words Q, R.

We have a surjective map $\rho : S \to k[GO_n^m]^{\mathrm{Ad}GO_n}$ defined by $\rho(U_Q) = \mathrm{tr}(Q)$ and $\rho(l_Q) = \lambda(Q)$.

Proposition 14. $\ker(\rho)$ *is the radical of the ideal generated by the relations:*

- $U_Q - l_QU_{Q^{-1}}$ *for* $Q \in \mathrm{FG}(\{A_1, \ldots, A_m\})$
- $G_{j,n+1}(Q_1, \ldots, Q_{n+1})$, $0 \le j \le (n+1)/2$, *as the* Q_i *vary over words in* $\mathrm{FG}(\{A_1, \ldots, A_m\})$, *defined as follows. First, define* $G'_{j,n+1}(X_1, \ldots, X_{n+1})$ *to be the same as* $F_{j,n+1}(X_1, \ldots, X_{n+1})$ *except that we replace each* T_M, $M \in \mathrm{FS}(\{X_1, X_1^t, \ldots, X_{n+1}, X_{n+1}^t\})$, *with* $l_{M'}U_{M'}$, *where* $M' \in \mathrm{FG}(\{X_1, \ldots, X_{n+1}\})$ *is the result of substituting all transposed letters* X_i^t *in* M *with* X_i^{-1}. *Then*

$$G_{j,n+1}(Q_1, \ldots, Q_{n+1}) = (G'_{j,n+1}(X_1, \ldots, X_{n+1}))\{X_i \mapsto Q_i\}.$$

Proof. Let $J \subset S$ be the ideal generated by relations of the form $U_Q - l_QU_{Q^{-1}}$. Then ρ induces a surjective map $\overline{\rho} : S/J \twoheadrightarrow k[GO_n^m]^{\mathrm{Ad}GO_n}$. Easily $\psi = \overline{\rho} \circ \tau$ where $\tau : R \to S/J$ is defined by: $\tau(T_M) = l_{M''}U_{M'}$, where M' is the result of substituting all transposed letters A_i^t in M with A_i^{-1}, and M'' is the product (with multiplicity) of all letters $A_1, \ldots, A_m$ that appear transposed in M. Then $\ker(\rho) = J + \ker(\overline{\rho}) = J + \tau(\ker(\psi))$.

The ideal J corresponds to the relations $U_Q - l_QU_{Q^{-1}}$. Applying τ to the relations $T_{NN^tP} - \frac{1}{n}T_{NN^t}T_P$ from Proposition 12 yields 0.

It remains to show that $\tau(\ker(\pi))$ is the ideal generated by the relations $G_{j,n+1}(Q_1, \ldots, Q_{n+1})$. Let $0 \le j \le (n+1)/2$ and let $M_1, \ldots, M_{n+1} \in \mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$, so that $F_{j,n+1}(M_1, \ldots, M_{n+1})$ is one of the generators of $\ker(\pi)$ in Theorem 11. Then $\tau(F_{j,n+1}(M_1, \ldots, M_{n+1}))$ is the same as $F_{j,n+1}(M_1, \ldots, M_{n+1})$ except that we replace each T_M with $l_{M''}U_{M'}$, where M' and M'' are as in the definition of τ. From the definition of $F_{j,n+1}$, in each monomial $T^j_\sigma(M_1, \ldots, M_{n+1})$ of $F_{j,n+1}(M_1, \ldots, M_{n+1})$, all subscripts of T are in $\mathrm{FS}(\{M_1, M_1^t, \ldots, M_{n+1}, M_{n+1}^t\})$ and every M_i appears exactly once (possibly transposed). Thus for each i, $\tau(T^\sigma_j(M_1, \ldots, M_{n+1}))$ gets a factor of either $l_{M_i''}$ or $l_{(M_i^t)''}$ depending on whether M_i or M_i^t appears. Now using the identities $l_{NP} = l_{PN}$ and $l_{N^{-1}} = l_N^{-1}$, easily $l_{(M_i^t)''}/l_{M_i''} = l_{M_i'}$, so

$$\tau(F_{j,n+1}(M_1, \ldots, M_{n+1}))/\prod_{i=1}^{n+1} l_{M_i''} = G_{j,n+1}(M_1', \ldots, M_{n+1}').$$

This proves the proposition as $\prod_{i=1}^{n+1} l_{M_i''}$ is an invertible element of S. □

From this proposition, we immediately deduce the following finite presentation up to radical of $k[GO_n^\bullet]^{\mathrm{Ad}GO_n}$ as an FFG-algebra.

Corollary 15. *Let $A^\bullet = F_{\mathrm{FFG}}(1) \otimes F_{\mathrm{FFG}}(1)$ be an* FFG*-algebra with two free generators of arity 1, denoted by T and l. Then the FFG-algebra map $\Theta^\bullet : A^\bullet \to k[GO_n^\bullet]^{\mathrm{Ad}GO_n}$ sending T to $\mathrm{tr}(A_1)$ and l to $\lambda(A_1)$ is surjective. For $g \in \mathrm{FG}(\{g_1, \ldots, g_n\})$, let ϕ_g denote some fixed map $\mathrm{FG}(\{g_1, \ldots, g_n\}) \to \mathrm{FG}(\{g_1, \ldots, g_n\})$ sending g_1 to g. Note that $\Theta^\bullet(A^{\phi_g}(T)) = \mathrm{tr}(g\{g_i \mapsto A_i\})$ and $\Theta^\bullet(A^{\phi_g}(l)) = \lambda(g\{g_i \mapsto A_i\})$. Then the kernel of $\Theta^\bullet$ is the radical of the FFG-ideal generated by the relations:*

- $A^{\phi_1}(T) - n$
- $A^{\phi_{g_1 g_2}}(T) - A^{\phi_{g_2 g_1}}(T)$
- $A^{\phi_1}(l) - 1$
- $A^{\phi_{g_1 g_2}}(l) - A^{\phi_{g_1}}(l) A^{\phi_{g_2}}(l)$
- $T - l A^{\phi_{g_1^{-1}}}(T)$
- $H_{j,n+1}(g_1, \ldots, g_{n+1})$, $0 \leq j \leq (n+1)/2$, *which we define to be the same as $G_{j,n+1}(g_1, \ldots, g_{n+1})$ except that we replace each variable l_g with $A^{\phi_g}(l)$ and each U_g with $A^{\phi_g}(T)$.*

We are now ready to define pseudocharacters for GO_n and O_n.

Definition 7. Let Γ be a group. A *GO_n-pseudocharacter of Γ over k* is a pair (T, l), consisting of a set map $T : \Gamma \to k$ and a group homomorphism $l : \Gamma \to k^\times$, such that

- $T(1) = n$
- For all $\gamma_1, \gamma_2 \in \Gamma$, $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$
- For all $\gamma \in \Gamma$, $T(\gamma) = l(\gamma)T(\gamma^{-1})$
- For all integers $0 \leq j \leq (n+1)/2$ and for all $\gamma_1, \ldots, \gamma_{n+1} \in \Gamma$, T and l satisfy the relation

  $$I_{j,n+1}(l, T, \gamma_1, \ldots, \gamma_{n+1}) = 0,$$

  where we define $I_{j,n+1}(l, T, \gamma_1, \ldots, \gamma_{n+1})$ to be the same as $G_{j,n+1}(\gamma_1, \ldots, \gamma_{n+1})$ except that we replace each variable l_γ with $l(\gamma)$ and each U_γ with $T(\gamma)$.

Definition 8. An *O_n-pseudocharacter of Γ over k* is a set map $T : \Gamma \to k$ such that $(T, 1)$ is a GO_n-pseudocharacter.

Theorem 16. *Assume k is a topological field of characteristic 0.*

(1) Let $\rho : \Gamma \to GO_n(k)$ be a continuous (with the k-topology on $GO_n(k)$) homomorphism. Then $(\mathrm{tr}(\rho), \lambda(\rho))$ is a GO_n-pseudocharacter.

(2) Conversely, let $\overline{k}$ have a topology extending the topology on k. Let (T, l) be a GO_n-pseudocharacter. Then there is a finite extension k' of k and a continuous semisimple homomorphism $\rho : \Gamma \to GO_n(k')$ such that $\mathrm{tr}(\rho) = T$ and $\lambda(\rho) = l$. Moreover, ρ is unique up to conjugation by $GO_n(\overline{k})$.

(3) Let $\rho : \Gamma \to GO_n(k)$ be a semisimple homomorphism. Then

$$\ker(\rho) = \{\eta \in \Gamma \mid \lambda(\eta) = 1 \text{ and } T(\gamma\eta) = T(\gamma) \text{ for all } \gamma \in \Gamma\}.$$

The same result holds with GO_n replaced by O_n and GO_n-pseudocharacters (T, l) replaced by O_n-pseudocharacters T.

Proof. This is a direct consequence of our general result on pseudocharacters (Theorem 5), using the finite presentation up to radical of $k[GO_n^\bullet]^{\mathrm{Ad}GO_n}$ described in Corollary 15. ☐

Remark 6. The above results can also be proven by modifying Taylor's proof for GL_n-pseudocharacters [Tay, Theorem 1]. In fact, one can generalize the above result to algebras, as follows. First define a $*$-algebra to be a (possibly noncommutative) $\overline{k}$-algebra with an involution $*$. Define an orthogonal n-dimensional representation of a $*$-algebra R to be a $\overline{k}$-algebra morphism $R \to M_n(\overline{k})$ mapping $*$ to the transpose. Then one can define n-dimensional orthogonal pseudocharacters of a $*$-algebra R similarly to the definition of O_n-pseudocharacters above. Using [Pro2, Theorem 15.3] in place of [Tay, Lemma 2] in Taylor's proof, one can prove that these are in bijection with $O_n(\overline{k})$-conjugacy classes of semisimple orthogonal representations of R. By taking R to be the group algebra $\overline{k}[\Gamma]$ with involution determined by $(\gamma)^* = l(g)(g^{-1})$ for $\gamma \in \Gamma$, one recovers Theorem 16.

2.2 (General) Symplectic Group

Again let k be a field of characteristic 0. Let $GSp_{2n}(k) = \{A \in M_{2n}(k) \mid \text{for some } \lambda \in k^\times,\ AA^* = \lambda I\}$ be the n-dimensional general symplectic group; here $*$ is the symplectic involution

$$A^* = \Omega^{-1} A^T \Omega$$

where

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the matrix of the standard symplectic form. It is a connected reductive algebraic group.

The results and proofs for GSp_{2n} are exactly analogous to those for GO_n, except that instead of starting with the relations $F_{j,n+1}$ defined above, we start with the relations $F^i_{h,n}$, for $1 \le i \le n+1$ and $0 \le h < i$, defined in [Pro2, Theorem 10.2(a)]. For convenience, we state the analogue of Theorem 16; from this and the original proof, it is easy to read off a finite presentation up to radical of $k[GSp_{2n}^\bullet]^{\mathrm{Ad}GSp_{2n}}$ as an FFG-algebra.

Define a function $\lambda : GSp_{2n}(k) \to k^\times$ by $AA^* = \lambda(A)I$. Note that $\lambda \in k[GSp_{2n}]^{\mathrm{Ad}GSp_{2n}}$.

Definition 9. Let Γ be a group. A *GSp_{2n}-pseudocharacter of* Γ *over* k is a pair (T, l), consisting of a set map $T : \Gamma \to k$ and a group homomorphism $l : \Gamma \to k^\times$, such that

- $T(1) = 2n$
- For all $\gamma_1, \gamma_2 \in \Gamma$, $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$
- For all $\gamma \in \Gamma$, $T(\gamma) = l(\gamma)T(\gamma^{-1})$
- For all integers $1 \le i \le n+1$ and $0 \le h < i$, and for all $\gamma_1, \ldots, \gamma_{n+i} \in \Gamma$, T and l satisfy the relation

$$I^i_{h,n+1}(l, T, \gamma_1, \ldots, \gamma_{n+i}) = 0,$$

  where $I^i_{h,n+1}(l, T, \gamma_1, \ldots, \gamma_{n+i})$ is defined as follows:

  – Taking $X_1, \ldots, X_{n+i}$ to be matrix variables, define $G^i_{h,n+1}(X_1, \ldots, X_{n+i})$ to be the same as $F^i_{h,n+1}(X_1, \ldots, X_{n+i})$, except that we replace each variable T_M with formal symbols $l_{M'}U_{M'}$, where $M' \in \mathrm{FG}(\{X_1, \ldots, X_{n+i}\})$ is the result of substituting all transposed letters X_j^t in M with X_j^{-1}.

  – Define $I^i_{h,n+1}(l, T, \gamma_1, \ldots, \gamma_{n+i})$ to be the same as $G^i_{h,n+1}(\gamma_1, \ldots, \gamma_{n+i})$ except that we replace each symbol l_γ with $l(\gamma)$ and each U_γ with $T(\gamma)$.

Definition 10. An *Sp_{2n}-pseudocharacter of* Γ *over* k is a set map $T : \Gamma \to k$ such that $(T, 1)$ is a GSp_{2n}-pseudocharacter.

Theorem 17. *Assume k is a topological field of characteristic 0.*

(1) Let $\rho : \Gamma \to GSp_{2n}(k)$ be a continuous (with the k-topology on $GSp_{2n}(k)$) homomorphism. Then $(\mathrm{tr}(\rho), \lambda(\rho))$ is a GSp_{2n}-pseudocharacter.

(2) Conversely, let $\overline{k}$ have a topology extending the topology on k. Let (T, l) be a GSp_{2n}-pseudocharacter. Then there is a finite extension k' of k and a continuous semisimple homomorphism $\rho : \Gamma \to GSp_{2n}(k')$ such that $\mathrm{tr}(\rho) = T$ and $\lambda(\rho) = l$. Moreover, ρ is unique up to conjugation by $GSp_{2n}(\overline{k})$.

(3) Let $\rho : \Gamma \to GSp_{2n}(k)$ be a semisimple homomorphism. Then

$$\ker(\rho) = \{\eta \in \Gamma \mid \lambda(\eta) = 1 \text{ and } T(\gamma\eta) = T(\gamma) \text{ for all } \gamma \in \Gamma\}\,.$$

The same result holds with GSp_{2n} replaced by Sp_{2n} and GSp_{2n}-pseudocharacters (T, l) replaced by Sp_{2n}-pseudocharacters T.

2.3 Special Orthogonal Group

Odd Dimension

When the dimension is $2n+1$ for some n, we have $k[GO^\bullet_{2n+1}]^{\mathrm{Ad}SO_{2n+1}} = k[GO^\bullet_{2n+1}]^{\mathrm{Ad}O_{2n+1}}$, since every orthogonal matrix is ± 1 times a special orthogonal matrix. By the same reasoning as in the proof of Proposition 10, this equals $k[GO^\bullet_{2n+1}]^{\mathrm{Ad}GO_{2n+1}}$. Next, the kernel of the natural surjective map $k[GO^\bullet_{2n+1}]^{\mathrm{Ad}SO_{2n+1}} \twoheadrightarrow k[SO^\bullet_{2n+1}]^{\mathrm{Ad}SO_{2n+1}}$ is generated up to radical by the relation $\det(A_1) - 1$ (expressed in terms of $\mathrm{tr}(A_1)$, $\mathrm{tr}(A_1^2)$, etc.). Hence $k[SO^\bullet_{2n+1}]^{\mathrm{Ad}SO_{2n+1}}$ is generated by tr as an FFG-algebra, and the relations between tr are generated, up to radical, by the relations for $k[GO^\bullet_{2n+1}]^{\mathrm{Ad}GO_{2n+1}}$ with $\lambda = 1$ and the relation det $= 1$ expressed in terms of tr.

Definition 11. An *(odd-dimensional) SO_{2n+1}-pseudocharacter of G over k* is an O_{2n+1}-pseudocharacter $T : G \to k$ which additionally satisfies the relation $\det(T)(g) = 1$ for all $g \in G$, where $\det(T)(g)$ is a polynomial in the $T(g^i)$ such that $\det(\mathrm{tr})(B) = \det(B)$ for all matrices B.

Then the usual result holds by our general result on pseudocharacters (Theorem 5) and the above discussion.

Even Dimension

When the dimension is $2n$ for some n, the invariant theory of SO_{2n} is more complicated. Aslaksen, Tan, and Zhu [ATZ, Theorem 3] show that for all m, $k[M^m_{2n}]^{\mathrm{Ad}SO_{2n}}$ is generated as a k-algebra by tr and the n-ary *linearized Pfaffian* pl, defined as the full polarization of the function

$$\widetilde{\mathrm{pf}}(W) = \mathrm{pf}(W - W^t)$$

where pf is the usual Pfaffian. Here the inputs to tr and pl are again drawn from $\mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$. Then $k[SO^\bullet_{2n}]^{\mathrm{Ad}SO_{2n}}$ is generated as an FFG-algebra by tr and pl.

A result due to Rogora [Rog] allows us to determine the relations between these generators up to radical, as follows.

Lemma 18. *The* FFG*-ideal of relations between the generators* tr *and* pl *of $k[SO^\bullet_{2n}]^{\mathrm{Ad}SO_{2n}}$ is the radical of the* FFG*-ideal generated by the relations between* tr *for $k[SO^\bullet_{2n}]^{\mathrm{Ad}GO_{2n}}$ and the relation described in [Rog, Theorem 3.2].*

Proof. Let R be a polynomial in terms of the given generators (i.e., in terms of their images under the internal morphisms in the free FFG-algebra) which maps to 0 in $k[SO^\bullet_{2n}]^{\mathrm{Ad}SO_{2n}}$. Note that conjugating all inputs to R by an element of $O_{2n}(k) \setminus SO_{2n}(k)$ preserves the value of any generator $\mathrm{tr}(M)$ or $\lambda(M)$ while negating the value of any generator $\mathrm{pl}(M_1, \ldots, M_n)$. Thus conjugating all inputs of any monomial in R sends that monomial to either itself or its negation; we call the monomial "even" in the former case and "odd" in the latter case. Let R_e and R_o be the sums of all even and odd monomials in R, respectively. Then R_e and $-R_o$ are mapped to the same image in $k[SO_{2n}^{\bullet}]^{\mathrm{Ad}SO_{2n}}$. Then conjugating all of their image's inputs by an element of $O_{2n}(k) \setminus SO_{2n}(k)$, we see that R_e and R_o also map to the same image in $k[SO_{2n}^{\bullet}]^{\mathrm{Ad}SO_{2n}}$. Hence R_e and R_o both map to 0, so that they are both in the FFG-ideal of relations.

It now suffices to show that the even and odd relations are in the given FFG-ideal. If R_e is an even relation, then each of its monomials consists of traces and pairs of linearized Pfaffians. After replacing each pair of linearized Pfaffians with a polynomial in traces using the relations described in [Rog, Theorem 3.2], we get a polynomial in terms of traces which is a GO_{2n}-invariant. Hence R_e is in the given FFG-ideal. Next, if R_o is an odd relation, then R_o^2 is an even relation, hence is in the given FFG-ideal. Then R_o is in the radical of the given FFG-ideal. □

Definition 12. An *(even-dimensional) SO_{2n}-pseudocharacter of G over k* is a pair of functions $T : G \to k$, $P : G^n \to k$, such that

- T is an O_{2n}-pseudocharacter of G over k
- For all $g \in G$, $\det(T)(g) = 1$
- For all $g_1, \ldots, g_n, h_1, \ldots, h_n \in G$, $P(g_1, \ldots, g_n)P(h_1, \ldots, h_n)$ satisfies the relation in [Rog, Theorem 3.2] with P in place of Q and T in place of tr.

Then we have the usual result.

3 Application: Conjugacy vs. Element-Conjugacy

In this section, we use our pseudocharacters to answer questions about conjugacy vs. element-conjugacy of group homomorphisms $\Gamma \to H(k)$ for H a linear algebraic group, following Larsen [Lar1, Lar2].

Definition 13. Fix a linear algebraic group H over a field k, and let Γ be an abstract group. Two homomorphisms $\rho_1, \rho_2 : \Gamma \to H(k)$ are called *globally conjugate* if there exists $h \in H(k)$ such that $\rho_1 = h\rho_2 h^{-1}$. They are called *element-conjugate* if for all $\gamma \in \Gamma$, there exists $h_\gamma \in H(k)$ such that $\rho_1(\gamma) = h_\gamma \rho_2(\gamma) h_\gamma^{-1}$.

Recall that we call a homomorphism $\rho : \Gamma \to H(k)$ *semisimple* if the Zariski closure of $\mathrm{Im}(\sigma)$ in $H(k)$ is reductive. The conjugacy vs. element-conjugacy question for $H(k)$ asks whether or not element-conjugate semisimple homomorphisms $\Gamma \to H(k)$ are automatically globally conjugate.

Definition 14. A linear algebraic group $H(k)$ is *acceptable* if element-conjugacy implies global conjugacy for all semisimple homomorphisms of arbitrary groups Γ. We call $H(k)$ *finite-acceptable* if element-conjugacy implies global conjugacy for all finite groups Γ, and *compact-acceptable* if k is topological and element-conjugacy implies conjugacy for all continuous semisimple homomorphisms of compact groups Γ.

In [Lar1, Lar2], Larsen mostly classifies the complex and compact simple Lie groups as finite-acceptable or finite-unacceptable (which implies unacceptable). Recent results by Fang, Han, and Sun [FHS] show that $GL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$, and their real compact forms are in fact compact-acceptable.

In this section, we give a simple sufficient condition for the acceptability of a connected reductive group H over an algebraically closed field k of characteristic 0, in terms of the FFG-algebra $k[H^{\bullet}]^{\mathrm{Ad}H}$. This condition and the results of Section 2 immediately imply that $GO_n(k)$, $O_n(k)$, $GSp_{2n}(k)$, $Sp_{2n}(k)$, and $SO_{2n+1}(k)$ are acceptable (not just finite- or compact-acceptable). By [Lar1, Proposition 1.7], it follows that the maximal compact subgroups of these groups over $\mathbb{C}$ are compact-acceptable. Previous results of this form were only known for $O_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$, $SO_{2n+1}(\mathbb{C})$, and $SO_4(\mathbb{C})$, and only for compact-acceptability.

We also use our pseudocharacters for SO_{2n} to give a criterion for when a semisimple homomorphism $\rho : \Gamma \to SO_{2n}(k)$ is a counterexample to acceptability for $SO_{2n}(k)$, at least when Γ is torsion.

Using this criterion, we prove that $SO_4(k)$ is acceptable, improving a result due to Yu [Yu1] showing that $SO_4(\mathbb{C})$ is compact-acceptable. We also construct a counterexample to acceptability for $SO_{2n}(k)$ ($n \geq 3$) with domain group $\Gamma = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$; this gives a simpler example than the one in [Lar1, Proposition 3.8], and it additionally shows that $SO_6(k)$ is unacceptable, a result which was not previously known.

3.1 General Principles

Let H be a linear algebraic group. Suppose that H has pseudocharacters consisting of arity 1 functions only. More formally, let k be an algebraically closed field of characteristic 0, and suppose that there exist invariants $f_1, \dots, f_n \in k[H]^{\mathrm{Ad}H}$ such that for any group Γ, the map $\rho \mapsto (f_1(\rho), \dots, f_n(\rho))$ induces a bijection between

$$\{H(k)\text{-conjugacy classes of semisimple homomorphisms } \rho : \Gamma \to H(k)\}$$

and

$$\{\text{maps } F_1, \dots, F_n : \Gamma \to k \text{ satisfying certain fixed relations}\}.$$

Then $H(k)$ is acceptable: indeed, if $\rho_1, \rho_2 : \Gamma \to H(k)$ are semisimple element-conjugate homomorphisms, then for all $\gamma \in \Gamma$, we have $f_i(\rho_1(\gamma)) = f_i(\rho_2(\gamma))$ for $1 \leq i \leq n$, so ρ_1 and ρ_2 have the same H-pseudocharacters, hence they are conjugate.

When H is a connected reductive group (or, more generally, when the conclusion of Theorem 5 holds for H, such as for $H = O_n$), we can restate this result as follows.

Theorem 19. *Let H be an algebraic group over an algebraically closed field k of characteristic 0 such that Theorem 5 holds for H with the action of* AdH *(e.g., H is a connected reductive group). Suppose that $k[H^\bullet]^{\mathrm{Ad}H}$ is generated by $k[H]^{\mathrm{Ad}H}$ as an* FFG*-algebra, i.e., by functions of arity 1. Then $H(k)$ is acceptable.*

In particular, $GO_n(k)$, $O_n(k)$, $GSp_{2n}(k)$, $Sp_{2n}(k)$, and $SO_{2n+1}(k)$ are acceptable.

Although it would be convenient if the converse to this theorem were true, it is not. We show below that $SO_4(k)$ is acceptable, while Yu [Yu2] has shown that the arity 2 function pl in $k[SO_4^\bullet]^{\mathrm{Ad}SO_4}$ is not generated by $k[SO_4]^{\mathrm{Ad}SO_4}$.

Even when H is not acceptable, so that a homomorphism can have element-conjugate but not conjugate homomorphisms, the number of such homomorphisms is uniformly bounded, as follows.[2]

Proposition 20. *Let H be a connected reductive algebraic group over an algebraically closed field k of characteristic 0. Then there exists $N_H \in \mathbb{N}$ such that for any semisimple homomorphism $\rho : \Gamma \to H(k)$, the number of $H(k)$-conjugacy classes of semisimple homomorphisms $H(k)$-element-conjugate to ρ is at most N_H.*

Proof. Let d be such that H is an affine subgroup variety of GL_d. Then any semisimple homomorphism $H(k)$-element-conjugate to ρ is also $GL_d(k)$-element-conjugate, hence $GL_d(k)$-conjugate. We will show that the number of $H(k)$-conjugacy classes of homomorphisms into $H(k)$ that are $GL_d(k)$-conjugate to ρ is bounded by some N_H.

Replacing Γ by the Zariski closure of $\mathrm{Im}(\rho)$, it suffices to prove the proposition when Γ is an algebraic group and we only consider homomorphisms that are also algebraic maps. By Lemma 7, there exists $q_d \in \mathbb{N}$ depending only on d such that $\Gamma = \overline{\langle \gamma_1, \dots, \gamma_{q_d} \rangle}$ for some $\gamma_1, \dots, \gamma_{q_d} \in \Gamma$.

If ρ' is such that $f(\rho(\gamma_1), \dots, \rho(\gamma_{q_d})) = f(\rho'(\gamma_1), \dots, \rho'(\gamma_{q_d}))$ for all $f \in k[H^{q_d}]^{\mathrm{Ad}H}$, then ρ' is H-conjugate to ρ, as follows. For any $m \in \mathbb{N}$, $g \in k[H^m]^{\mathrm{Ad}H}$, and $\eta_1, \dots, \eta_m \in \langle \gamma_1, \dots, \gamma_{q_d} \rangle$, there is some $\hat{g} \in k[H^{q_d}]^{\mathrm{Ad}H}$ such that $g(\rho(\eta_1), \dots, \rho(\eta_m)) = \hat{g}(\rho(\gamma_1), \dots, \rho(\gamma_{q_d}))$ and likewise for ρ', hence $g(\rho(\eta_1), \dots, \rho(\eta_m)) = g(\rho'(\eta_1), \dots, \rho'(\eta_m))$. Thus the H-pseudocharacters of ρ and ρ' are identical on $\langle \gamma_1, \dots, \gamma_{q_d} \rangle$, from which they are identical on all of Γ. Then ρ' is $H(k)$-conjugate to ρ by the uniqeness claim in Theorem 5(2).

[2] I thank one of the anonymous reviewers for bringing to my attention this question and its relation to [Vin].

Now by a result of Vinberg [Vin, Theorem 1], the natural map $\mathrm{Spec}(k[H^{q_d}]^{\mathrm{Ad}H}) \to \mathrm{Spec}(k[GL_d^{q_d}]^{\mathrm{Ad}GL_d})$ is finite. Hence for a semisimple algebraic homomorphism ρ' that is $GL_d(k)$-conjugate to ρ, the number of possible values for the $f(\rho'(\gamma_1), \ldots, \rho'(\gamma_{q_d}))$, $f \in k[H^{q_d}]^{\mathrm{Ad}H}$, is bounded by a constant depending only on H.

□

3.2 Element-conjugacy vs. Conjugacy for SO_{2n}

Let k be an algebraically closed field of characteristic 0, and let n be an integer. We wish to characterize all pairs of semisimple homomorphisms $\rho_1, \rho_2 : \Gamma \to SO_{2n}(k)$ that are element-conjugate but not globally conjugate, at least when Γ is torsion. Let pl denote the linearized antisymmetrized Pfaffian (see Section 2.3 above). Our first result is as follows.

Proposition 21. *Let Γ be a group, and let $\rho_1 : \Gamma \to SO_{2n}(k)$ be a semisimple homomorphism. If there exists a semisimple homomorphism $\rho_2 : \Gamma \to SO_{2n}(k)$ that is element-conjugate but not globally conjugate to ρ_1, then:*

- *For all $\gamma \in \Gamma$, $\mathrm{pf}(\rho_1(\gamma) - \rho_1(\gamma)^t) = 0$*
- *There exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\mathrm{pl}(\rho_1(\gamma_1), \ldots, \rho_1(\gamma_n)) \neq 0$.*

If Γ is torsion, then the converse holds as well.

When such a ρ_2 exists, it is unique up to conjugation by $SO_{2n}(k)$, and it is given by

$$\rho_2 = X \rho_1 X^{-1}$$

for some $X \in O_{2n}(k) \setminus SO_{2n}(k)$.

Proof. **Uniqueness:** Let ρ_2 be element-conjugate but not globally conjugate to ρ_1 in SO_{2n}. Then ρ_1 and ρ_2 are element-conjugate in O_{2n}, hence globally conjugate in O_{2n}. Thus there is an $X \in O_{2n}(k)$ such that $\rho_2 = X\rho_1 X^{-1}$, and necessarily $X \notin SO_{2n}(k)$. Since $SO_{2n}(k)$ has index 2 in $O_{2n}(k)$, any other choice of X gives a homomorphism that is globally conjugate to ρ_2 in $SO_{2n}(k)$.

Existence, ($\Longrightarrow$): Let ρ_2 be a semisimple homomorphism that is element-conjugate but not globally conjugate to ρ_1. The invariant pl is an "odd" invariant in the sense that

$$\begin{aligned} \mathrm{pl}(\rho_1(\gamma_1), \ldots, \rho_1(\gamma_n)) &= -\mathrm{pl}(X\rho_1(\gamma_1)X^{-1}, \ldots, X\rho_1(\gamma_n)X^{-1}) \\ &= -\mathrm{pl}(\rho_2(\gamma_1), \ldots, \rho_2(\gamma_n)) \end{aligned} \tag{2}$$

for all $\gamma_1, \ldots, \gamma_n \in \Gamma$. Since ρ_1 and ρ_2 are not globally conjugate, they must have different pseudocharacters, and since $\mathrm{tr}(\rho_1) = \mathrm{tr}(\rho_2)$ by element-conjugacy, there must exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that

$$\mathrm{pl}(\rho_1(\gamma_1), \ldots, \rho_1(\gamma_n)) \neq \mathrm{pl}(\rho_2(\gamma_1), \ldots, \rho_2(\gamma_n)).$$

Then by (2), $\mathrm{pl}(\rho_1(\gamma_1), \ldots, \rho_1(\gamma_n)) \neq 0$.

Next, since ρ_1 and ρ_2 are element-conjugate, $\rho_1|_{\langle\gamma\rangle}$ is SO_{2n}-conjugate to $\rho_2|_{\langle\gamma\rangle}$ for each $\gamma \in \Gamma$, so

$$\mathrm{pl}(\rho_1(\gamma^{m_1}), \ldots, \rho_1(\gamma^{m_n})) = \mathrm{pl}(\rho_2(\gamma^{m_1}), \ldots, \rho_2(\gamma^{m_n}))$$

for all $\gamma \in \Gamma$ and $m_1, \ldots, m_n \in \mathbb{Z}$. Then by (2), $\mathrm{pl}(\rho_1(\gamma^{m_1}), \ldots, \rho_1(\gamma^{m_n})) = 0$. In particular, $\widetilde{\mathrm{pf}}(\rho_1(\gamma)) = \frac{1}{n!}\mathrm{pl}(\rho_1(\gamma), \ldots, \rho_1(\gamma)) = 0$ for all $\gamma \in \Gamma$. Hence

$$\mathrm{pf}(\rho_1(\gamma) - \rho_1(\gamma)^t) = \widetilde{\mathrm{pf}}(\rho_1(\gamma)) = 0.$$

Existence, ($\Longleftarrow$): Assume Γ is torsion. Let $X \in O_{2n}(k) \setminus SO_{2n}(k)$, and set $\rho_2(\gamma) = X\rho_1(\gamma)X^{-1}$. Then by assumption, there exist $\gamma_1, \ldots, \gamma_n$ such that

$$\mathrm{pl}(\rho_1(\gamma_1), \ldots, \rho_1(\gamma_n)) \neq -\mathrm{pl}(\rho_1(\gamma_1), \ldots, \rho_1(\gamma_n)) = \mathrm{pl}(\rho_2(\gamma_1), \ldots, \rho_2(\gamma_n)),$$

so ρ_1 and ρ_2 are not globally conjugate.

Now fix $\gamma \in \Gamma$. Since Γ is torsion, Maschke's theorem implies that both $\rho_1|_{\langle\gamma\rangle}$ and $\rho_2|_{\langle\gamma\rangle}$ are semisimple. Thus to show that $\rho_1|_{\langle\gamma\rangle}$ and $\rho_2|_{\langle\gamma\rangle}$ are conjugate in SO_{2n}, it suffices to show that they have the same SO_{2n}-pseudocharacters. They have the same traces because ρ_1 and ρ_2 are conjugate in O_{2n}. To show that they have the same values of pl, we must show

$$\mathrm{pl}(\rho_1(\gamma^{m_1}), \dots, \rho_1(\gamma^{m_n})) = 0$$

for all $m_1, \dots, m_n \in \mathbb{Z}$, since the corresponding value for ρ_2 is the negative of that for ρ_1. By definition, $\mathrm{pl}(\rho_1(\gamma^{m_1}), \dots, \rho_1(\gamma^{m_n}))$ is the multilinear term in

$$\begin{aligned}&\widetilde{\mathrm{pf}}\left(t_1\rho_1(\gamma^{m_1}) + \cdots + t_n\rho_1(\gamma^{m_n})\right)\\ &= \mathrm{pf}\left(t_1(\rho_1(\gamma^{m_1}) - \rho_1(\gamma^{m_1})^t) + \cdots + t_n(\rho_1(\gamma^{m_n}) - \rho_1(\gamma^{m_n})^t)\right).\end{aligned}$$

But $\rho_1(\gamma) - \rho_1(\gamma)^t = \rho_1(\gamma) - \rho_1(\gamma)^{-1}$ divides $\rho_1(\gamma)^{m_i} - \rho_1(\gamma)^{-m_i} = \rho_1(\gamma^{m_i}) - \rho_1(\gamma^{m_i})^t$ for all i, so the assumption $\det(\rho_1(\gamma) - \rho_1(\gamma)^t) = \mathrm{pf}(\rho_1(\gamma) - \rho_1(\gamma)^t)^2 = 0$ implies that

$$\det\left(t_1(\rho_1(\gamma^{m_1}) - \rho_1(\gamma^{m_1})^t) + \cdots + t_n(\rho_1(\gamma^{m_n}) - \rho_1(\gamma^{m_n})^t)\right) = 0.$$

Hence taking the square root, the Pfaffian is zero as well for all values of $t_1, \dots, t_n$. Thus $\mathrm{pl}(\rho_1(\gamma^{m_1}), \dots, \rho_1(\gamma^{m_n})) = 0$, proving the proposition. □

We now use this result to determine the acceptability or unacceptability of $SO_{2n}(k)$ for all n. Previous results are as follows:

- $SO_2(k)$ is acceptable because it is abelian.
- $SO_4(\mathbb{C})$ is compact-acceptable. Yu [Yu1, Theorem 4.1(3)] recently showed that $SO_4(\mathbb{R})$ is compact-acceptable, so $SO_4(\mathbb{C})$ is compact-acceptable by [Lar1, Proposition 1.7]. Yu's proof uses the notion of strongly controlling fusion to show that the exceptional Lie group $\mathrm{G}_2(\mathbb{R})$ is compact-acceptable and then derives compact-acceptability of $SO_4(\mathbb{R})$ as a consequence.
- $SO_{2n}(\mathbb{C})$ is unacceptable for $n \geq 4$. Larsen [Lar1, Proposition 3.8] shows this by constructing a counterexample with domain group $\Gamma = SL_3(\mathbb{Z}/2\mathbb{Z})$.

We complete this program by proving that $SO_4(k)$ is acceptable and $SO_6(k)$ is unacceptable. Our counterexample to the acceptability of $SO_6(k)$, which extends to a counterexample to the acceptability of $SO_{2n}(k)$ for all $n \geq 3$, is especially simple, with $\Gamma = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Theorem 22. *$SO_4(k)$ is acceptable.*

Proof. Let $Q_1, Q_2 \in SO_4(k)$. We claim that if $\det(Q - Q^t) = 0$ for all $Q \in \langle Q_1, Q_2\rangle$, then $\mathrm{pl}(Q_1, Q_2) = 0$. The theorem then follows from the first statement in Proposition 21.

It suffices to prove the claim when $k = \mathbb{C}$. To simplify our computations, we use a variant of the special isomorphism $(SL_2(\mathbb{C}) \times SL_2(\mathbb{C}))/\langle(-I, -I)\rangle \cong SO_4(\mathbb{C})$ corresponding to the isoclinic decomposition of 4-dimensional rotations. Let $\mathbb{H}_\mathbb{C} = \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ denote the complex quaternions, and let $(\mathbb{H}_\mathbb{C})^* \cong SL_2(\mathbb{C})$ denote its unit group. For $q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \in \mathbb{H}_\mathbb{C}$, left (resp. right) multiplication by q on $\mathbb{H}_\mathbb{C}$ defines matrices $L(q)$ (resp. $R(q)$) given by

$$L(q) = \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & -q_4 & q_3 \\ q_3 & q_4 & q_1 & -q_2 \\ q_4 & -q_3 & q_2 & q_1 \end{pmatrix} \qquad R(q) = \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{pmatrix}.$$

Then we have an isomorphism

$$\varphi : \left((\mathbb{H}_\mathbb{C})^* \times (\mathbb{H}_\mathbb{C})^*\right)/\langle(-1,-1)\rangle \xrightarrow{\sim} SO_4(\mathbb{C})$$

given by $\varphi(q,r) = L(q)R(r)$.

Write $Q_1 = \varphi(u,v)$, $Q_2 = \varphi(w,x)$, and define $e = u_2w_2+u_3w_3+u_4w_4$, $f = v_2x_2+v_3x_3+v_4x_4$. One computes $\mathrm{pf}(\varphi(q,r)-\varphi(q,r)^t)/4 = r_1^2 - q_1^2$. Then since $Q_1Q_2 = \varphi(uw, vx)$ and $Q_1^2Q_2 = \varphi(u^2w, v^2x)$,

$$\begin{aligned}
\mathrm{pf}(Q_1 - Q_1^t)/4 &= v_1^2 - u_1^2 \\
\mathrm{pf}(Q_2 - Q_2^t)/4 &= x_1^2 - w_1^2 \\
\mathrm{pf}(Q_1Q_2 - (Q_1Q_2)^t)/4 &= (v_1x_1 - f)^2 - (u_1w_1 - e)^2 \\
\mathrm{pf}(Q_1^2Q_2 - (Q_1^2Q_2)^t)/4 &= (2v_1^2x_1 - 2v_1f - x_1)^2 - (2u_1^2w_1 - 2u_1e - w_1)^2.
\end{aligned}$$

By assumption, all of these Pfaffians are 0. It remains to show that $\mathrm{pl}(Q_1,Q_2)/8 = v_1x_1e - u_1w_1f$ is also 0. Applying the relations $x_1^2 = w_1^2$ and $(v_1x_1 - f)^2 = (u_1w_1 - e)^2$ to the relation, we get

$$x_1(v_1^2x_1 - v_1f) = w_1(u_1^2w_1 - u_1e).$$

Then $v_1x_1f = u_1w_1e$ because $x_1^2 = w_1^2$ and $u_1^2 = v_1^2$. Hence $x_1 = \pm w_1$, $u_1 = \pm v_1$, and $v_1x_1f = u_1w_1e$. Regardless of the choice of signs, $v_1x_1e = u_1w_1f$, proving the claim. □

Lemma 23. *$SO_6(k)$ is unacceptable.*

Proof. Let $\Gamma = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, with generators $(1,0)$ and $(0,1)$. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SO_2(k).$$

Define a homomorphism $\rho_6 : \Gamma \to SO_6(k)$ by

$$\begin{aligned}
\rho_6(1,0) &= A \oplus A \oplus I, \\
\rho_6(0,1) &= I \oplus A \oplus A.
\end{aligned}$$

Then one can check that $\det(\rho_6(\gamma) - \rho_6(\gamma)^t) = 0$ for all $\gamma \in \Gamma$, while

$$\mathrm{pl}(\rho_6(1,0), \rho_6(0,1), \rho_6(0,1)) = 16.$$

Hence by Proposition 21, ρ_6 is a counterexample to acceptability for $SO_6(k)$. □

More generally, we have:

Theorem 24. *Let Γ and A be as in the proof of the previous lemma. For any $n \geq 3$, the homomorphism $\rho_{2n} : \Gamma \to SO_{2n}(k)$ defined by*

$$\begin{aligned}
\rho_{2n}(1,0) &= A \oplus A \oplus I \oplus \bigoplus_{i=4}^{n} A, \\
\rho_{2n}(0,1) &= I \oplus A \oplus A \oplus \bigoplus_{i=4}^{n} A
\end{aligned}$$

satisfies $\det(\rho_{2n}(\gamma) - \rho_{2n}(\gamma)^t) = 0$ for all $\gamma \in \Gamma$ and $\mathrm{pl}(\rho_{2n}(1,0), \rho_{2n}(0,1), \ldots, \rho_{2n}(0,1)) \neq 0$. Hence by Proposition 21, ρ_{2n} gives a counterexample to acceptability for $SO_{2n}(k)$.

Proof. Let $\gamma \in \Gamma$, and write $\rho(\gamma) = \bigoplus_{i=1}^{n} B^{(i)}$. We have

$$\begin{aligned}
\det\left(\rho(\gamma) - \rho(\gamma)^t\right) &= \det\left(\bigoplus_{i=1}^{n}(B^{(i)} - (B^{(i)})^t)\right) \\
&= \prod_{i=1}^{n} \det(B^{(i)} - (B^{(i)})^t).
\end{aligned}$$

Hence to show $\det(\rho(\gamma) - \rho(\gamma)^t) = 0$, it suffices to prove that some 2×2 diagonal block $B^{(i)}$ of $\rho(\gamma)$ satisfies $\det(B^{(i)} - (B^{(i)})^t) = 0$. But one can check that for all $\gamma \in \Gamma$, one of the first three 2×2 diagonal blocks is $\pm I$.

Next, recall that for matrices $C_1, \ldots, C_n$, $\mathrm{pl}(C_1, \ldots, C_n)$ is defined to be the coefficient of $t_1 \cdots t_n$ in $\mathrm{pf}(t_1(C_1 - C_1^t) + \cdots + t_n(C_n - C_n^t))$. Letting each $C_j = \bigoplus_{i=1}^n C_j^{(i)}$ for some 2×2 matrices $C_j^{(i)}$, we have

$$\mathrm{pf}(t_1(C_1 - C_1^t) + \cdots + t_n(C_n - C_n^t)) = \prod_{i=1}^n \mathrm{pf}(t_1(C_1^{(i)} - (C_1^{(i)})^t) + \cdots + t_1(C_n^{(i)} - (C_n^{(i)})^t).$$

Now pf is a linear function of 2×2 antisymmetric matrices, so this equals

$$\prod_{i=1}^n \sum_{j=1}^n t_j \mathrm{pf}(C_j^{(i)} - (C_j^{(i)})^t).$$

Taking the coefficient of $t_1 \cdots t_n$ in this formula, we find that

$$\mathrm{pl}(C_1, \ldots, C_n) = \sum_{\sigma \in S_n} \prod_{i=1}^n \mathrm{pf}(C_{\sigma(i)}^{(i)} - (C_{\sigma(i)}^{(i)})^t).$$

Finally, note that $\mathrm{pf}(A - A^t) = 2$ and $\mathrm{pf}(I - I^t) = 0$. Thus

$$\mathrm{pl}\left(D_1 = \rho_{2n}(1, 0), D_2 = \rho_{2n}(0, 1), \ldots, D_n = \rho_{2n}(0, 1)\right)$$

will be positive so long as for some $\sigma \in S_n$, for all i, $D_{\sigma(i)}^{(i)} = A$. Taking σ to be the identity permutation works. $\square$

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